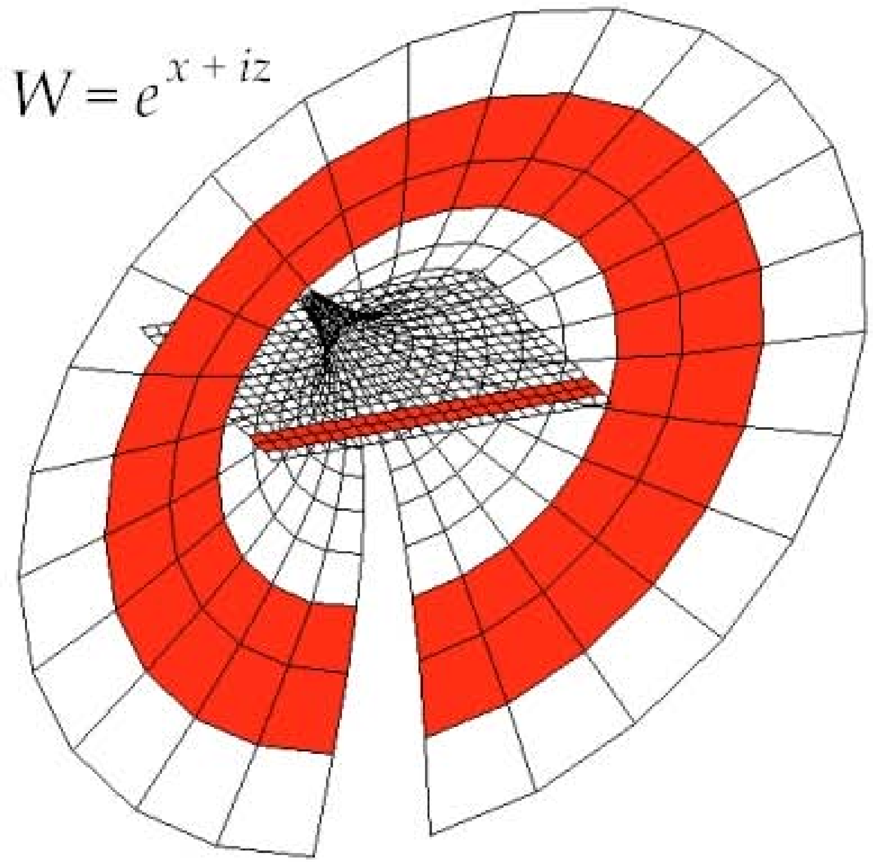
FOUNDATIONS OF TRANSCOMPLEX NUMBERS

An extension of the complex number system to four dimensions



Ernesto Pérez

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Preface

In general, the idea of a fourth dimension seems to precipitate authors into orgies of occultist mystification, rather than to lead to clear-sighted mathematical inquiry.

Rudolf v. B. Rucker
Geometry, Relativity and the Fourth Dimension

Why transcomplex numbers?

This book is about a generalization of the 2-dimension complex number system to a generalized 4-dimension system of complex numbers here called the Transcomplex Number System. As a consequence, it is also about an extension of the Cartesian coordinate system to a new hyperspace, the Transcomplex Space S^4, to represent transcomplex functions and variables.

I am convinced that the graphs of complex functions are not adequately represented on the complex Cartesian coordinates. When complex functions are plotted using this coordinate system the resulting graph tells very little about the true function behavior. Although we know that the real functions are special cases of other complex functions, the graphs of real functions are far from being—visually we mean—a special case from its complex counterpart. In fact, complex functions need not have resemblance with the more humble original real function graph, but intuition tell us that in the same way that the real numbers come out when the imaginary part of the complex numbers are chosen to be zero, so the graphs of the real functions should also come out when it is equated to zero the imaginary axis of the complex graph of a complex function. But this is not so, and good examples of this are the plotting of the complex logarithmic function and the complex trigonometric functions. The core of this behavior lies in the fact that the plotting of complex functions usually involves two separate complex planes: one for the domain of the function and the other for its range. But when we plot real functions we do not use two separate axes: we use intersecting orthogonal coplanar axes, both part of the same plane.

The intention of this book is to expose an alternative way to handle complex plotting and manipulations in a more intuitive–and hopefully elegant–way. To achieve this, the reader is only expected to be familiar with the basics of the set theory, the real and complex number system, and some elements about the plotting of complex functions. The mathematical symbolism of set theory will be kept to a minimum; however, its most elementary concepts will be sparingly used. The book is definition and theorem-oriented, that is, emphasis is given to the foundations of the transcomplex number system and in theorem proving. This is because theorems are proved using the definitions previously accepted, so that if the definitions are already clearly stated, the making and proving of theorems is greatly facilitated. Of course, it is in the proving of theorems that inconsistencies in a mathematical model are discovered, but, as will be shown later, the Transcomplex Number System and the complex numbers have a similar structure, thus it is very easy to state a theorem for the transcomplexs by extending an existing theorem from complex number field.

A reasonable but easy-going rigor is maintained throughout the book. Definitions and theorems are clearly labeled and relevant results are formulated as theorems.

It is in the final chapters devoted to surfaces of elementary transcomplex functions and applications of transcomplex numbers that the true implications of the Foundations are fully appreciated. The new point of view toward the plotting of complex variables will bring new figures not found in current mathematics textbooks and references about complex variables.

The applications to be work out are varied and chosen from complex variables textbooks to show how plotting of complex functions on the transcomplex domain are more elegant while the viewpoint is more general in the Transcomplex Number System.

Complex numbers are two-component numbers and for that reason we speak about the complex "plane". On the other hand, transcomplex numbers are four-element entities, and that makes them easy to associate to four-dimensioned spaces.

The reputed Hamilton's quaternions are also 4-entry mathematical constructs, but quaternions are not complex numbers, and one of its most notably characteristic is its noncommutativty of the multiplication operation. However, in this ground, the behavior of the transcomplexs is totally the same as the complex numbers because the transcomplexs are the same complex numbers, but extended to four dimensions.

But to extend the complexs to four dimensions and still retain its two-dimension structure and properties required to add two new types of numbers to the already known reals and imaginaries: the image-reals and the image-imaginaries. The image-real is a type of number that is is neither real nor imaginary, and the image-imaginary is the type of number that is neither real, nor image-real, nor imaginary.

One striking outcome of the transcomplex numbers is that in some cases the result of the multiplication of two of them can be zero even when none of them is zero; however when one of the factors is zero the result is a guaranteed zero.

This book is about a new mathematical structure with four “dimensions”: the transcomplex numbers. It is not about “the” fourth dimension, because one “fourth” dimension as such is useless. What is useful is a complete space with (not “in”) four dimensions.

The common way of referring to a “4-dimension space” is by visualizing it as repeating the real axis four times. But the approach of the transcomplex numbers is of four axes, each one with different properties; no one is identical to the other three. This results in handling the “space” properties in a novel –and hopefully– useful way.

Ernesto Pérez

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Chapter 1

Ordered Pairs

1.1 Introduction

This chapter is devoted to two basic concepts of the foundations of Transcomplex Numbers: the concept of Ordered Pair, and the concept of Transreal Number.

In order to have a consistent plane of coordinates for later expansion into a hyperspace to plot transcomplex number variables, a solid justification of its coordinates is needed. For its accomplishment, we need to build a solid foundation of the concept of ordered pair. Later, that concept will be expanded into the final concept of ordered pairs of ordered pairs.

Here, the addition, multiplication, exponentiation, and division of the ordered pairs are rigorously defined. Other concepts like absolute value and the argument of a real number ordered pairs are also introduced in this chapter. The introduction of the ordered pairs field will lead us into the concept of image-real numbers, a different entity when compared to the imaginary numbers, because the imaginary numbers are not closed under multiplication while the image-real real numbers do have this property.

Also new is the approach toward the re-definition of the Y-axis, and the definitions of perpendicularity of ordered pairs and orthogonality of axes. The X-axis does not merely consist of plain real numbers, but must be viewed as a special case of ordered pairs. The $Y^{\sim}$-axis is not the X-axis rotated 90°. The $Y^{\sim}$-axis does not consist of real numbers —neither of imaginary numbers. The $Y^{\sim}$-axis do have its own justified existence when viewed in terms of ordered pairs.

The definitions introduced for the operations with ordered pairs will bring some surprises, but later we will see the importance of those definitions and how all of them contribute to a final unified theory toward a four-dimensional space. For example, the absolute value of a real numbers ordered pair is not a single positive real number, but another ordered pair again, and the notion of norm of real numbers ordered pair is the one that replaces the usual concept of absolute value.

Finally, an important theorem for our *Foundations* proves that the ordered pairs system—in the way used and defined here—makes a consistent and closed field.

1.2 The real number system

1.2.1 The field of real numbers

Definition 1. Let $\mathbb{R}$ denote **the class of all real numbers**.

Associated with the real numbers are the two operations of addition and multiplication in the way it is used in the everyday mathematics. We are familiar with the commutativity of addition and multiplication; to the inconsistency of division by zero, etc. The aggregate of all those properties, together with the operations used on them, is called a field.

Formally, a field is defined as follows:

Definition 2. A **field**, denoted by $\{F; +, *\}$, is a set F of two or more elements, and two operations on it, **addition, +** ; and **multiplication, ***, that satisfy the following conditions:

1. The operation + is **closed** on F: if a and b are elements of F, then $a + b$ also belongs to F. In symbols:
$$a, b \in F \Rightarrow a + b \in F. \tag{1.2.1}$$

2. The operation + is **associative**: for any three elements a, b, and c of F
$$(a + b) + c = a + (b + c). \tag{1.2.2}$$

3. The operation + is **commutative**: for any two elements a and b of F
$$a + b = b + a. \tag{1.2.3}$$

4. There is an element 0 of F called the **identity unit of addition** or the **neutral element under addition** such that for any element a of F
$$0 + a = a. \tag{1.2.4}$$

5. For every element a of F the is an element -a on F called the **negative** of a such that
$$a + (-a) = 0. \tag{1.2.5}$$

6. The operation * is **closed** on F: if a and b are elements of F, then $a * b$ also belongs to F. In symbols:

$$a, b \in F \Rightarrow a * b \in F. \tag{1.2.6}$$

7. The operation $*$ is **associative**: for any three elements a, b, and c of F

$$(a * b) * c = a * (b * c). \tag{1.2.7}$$

8. The operation * is **commutative**: for any two elements a and b of F

$$a * b = b * a. \tag{1.2.8}$$

9. There is an element e of F called the **identity unit of multiplication** or the **neutral element under multiplication** such that for any element a of F

$$e * a = a. \tag{1.2.9}$$

10. For each element a other than 0 there is an a^{-1} called the **inverse** of a such that

$$a * a^{-1} = e. \tag{1.2.10}$$

11. The operation $*$ is **distributive** respect to addition, in symbols:

$$a * (b + c) = a * b + a * c. \tag{1.2.11}$$

12. For any two elements a and b of F

$$\text{if} \qquad a * b = 0 \qquad \text{then} \qquad a = 0 \qquad \text{or} \qquad b = 0. \tag{1.2.12}$$

1.2.2 Exponentiation and radicalization of real numbers

In addition to the properties stated for the real numbers, the exponentiation operations are essential for the practical usage of them.

Definition 3. The **exponentiation rules for the real numbers** are:.

$$a^0 = 1 \tag{1.2.13}$$
$$a^1 = a \tag{1.2.14}$$
$$a^n = a * a * a * \ldots * a \qquad n - \text{times} \tag{1.2.15}$$
$$a^{n+1} = a^n * a \tag{1.2.16}$$
$$a^n * a^m = a^{n+m} \tag{1.2.17}$$
$$(a^n)^m = a^{n*m} \tag{1.2.18}$$
$$(a^{-n}) = \frac{1}{a^n} \qquad \text{if} \quad a \neq 0 \tag{1.2.19}$$
$$\frac{a^n}{a^m} = a^{n-m} \qquad \text{if} \quad a \neq 0 \tag{1.2.20}$$
$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \qquad \text{if} \quad b \neq 0 \tag{1.2.21}$$
$$\sqrt[n]{a} = r \qquad \text{if} \quad r^n = a \tag{1.2.22}$$
$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad \text{if} \quad n \neq 0 \tag{1.2.23}$$
$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{(a^m)}. \tag{1.2.24}$$

1.2.3 Absolute Value of real numbers

Definition 4. The **absolute value** of a real number a is denoted by $|a|$ and is defined by the rule:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases} \tag{1.2.25}$$

1.2.4 Order relations

Definition 5. The **trichotomy** property. For any three real numbers a, b, and c, only one of the following relations holds:

$$a = b \qquad a > b, \qquad \text{or} \qquad a < b. \tag{1.2.26}$$

Definition 6. The **transitivity** property. For any three real numbers a, b, and c:

$$\text{if} \quad a > b \quad \text{and} \quad b > c \quad \text{then} \quad a > c. \tag{1.2.27}$$

1.3 Ordered Pairs

1.3.1 Ordered Pairs

Given two objects α and β, the set theory does guarantee, by virtue of its axioms, the existence of two more intuitive objects called ordered pairs of the objects α and β.

Definition 7. The **ordered pair** generated by the objects α and β, taken in that order, and denoted by the symbol $(\alpha\ ,\ \beta)$ is in itself different form both objects α and β, and is formally defined as follows:

$$(\alpha, \beta) = \Big\{\{\alpha\}, \{\alpha, \beta\}\Big\}. \tag{1.3.1}$$

Definition 8. Let the symbol $\mathbb{O}$ denote the **class of all ordered pairs**.

By the equality for sets, it is deduced that for two ordered pairs

$$P = (a, b) \qquad \text{and} \qquad P' = (a', b'). \tag{1.3.2}$$

$$P = P' \qquad \text{if and only if} \qquad a = a' \qquad \text{and} \qquad b = b'. \tag{1.3.3}$$

An ordered pair is a set of two elements, but not every set of two elements is an ordered pair. Thus, the set {a, b} is unordered because

$$\{a, b\} = \{b, a\}. \tag{1.3.4}$$

1.3.2 Real numbers ordered pairs

Definition 9. When a and b are real numbers, then we say that $(a,\ b)$ is a **real numbers ordered pair**, or an **ordered pair of real numbers**. Associated with the ordered pair $(a,\ b)$ is the ordered pair $(b,\ a)$ which we will call the **image** of the ordered pair $(a,\ b)$, or simply, the **image-ordered pair** $(b,\ a)$. A tilde, $\tilde{}$, above the symbol will denote the concept of image. Thus,

$$(a, b)^{\sim} = (b, a). \tag{1.3.5}$$

Note that from the above definition, it follows that

$$\big((a, b)^{\sim}\big)^{\sim} = (b, a)^{\sim} = (a, b). \tag{1.3.6}$$

1.3.3 Image-real numbers (transreal numbers)

Definition 10. The ordered pair $(a, 0)$ is **the real number** a, and the ordered pair $(0, a)$ is the **image-real number** $a^{\sim}$, or the **transreal number** $a^{\sim}$.

$$a = (a, 0) \qquad \text{and} \qquad a^{\sim} = (0, a). \tag{1.3.7}$$

The **zero real number** $(0, 0)$ is at the same time the **zero image-real number**. That is:

$$(0, 0) = (0, 0)^{\sim} = 0. \tag{1.3.8}$$

Definition 11. The **class of all image-real numbers** will be denoted by $\mathbb{R}^{\sim}$.

Definition 12. Ordered pairs with none of its components being equal zero will be called **non-axial ordered pairs**.

Image-reals are also positive or negative depending on the sign of the non-zero element.

1.3.4 The coordinate plane

Intuitively, we think of the real numbers as points on the "horizontal" X-axis, and we think of the image-reals as points on the "vertical" axis in the same coordinate system. However, in the true spirit of the mathematical definitions already stated, we cannot speak of that "vertical" axis as a 90° rotation of the X-axis, using the origin or zero-point as pivot, because rotation is not a predefined mathematical operation that can be executed on the field of the real numbers.

There is no need to think of the Y-axis as a "rotation" of the X-axis in order to justify its existence. The Y-axis is fully justified when the X-axis is accepted as a class of ordered pairs. In that case, the X- and Y-axes belong to the same class, the class $\mathbb{O}$ of all real numbers ordered pairs, making it useless to rely on the rotation concept.

The emphasis in the distinction between reals and image-reals is because, ordinarily, a single real number x on the Y-axis is considered as if it were the same real number x on the X-axis, but "rotated". The ordered pairs $(x, 0)$ and $(0, x)$ are different ordered pairs (when $x \neq 0$), so in the true spirit of the set theory and ordered pairs, the X and Y-axes are not the same entities.

Since our new Y-axis will consist of image-numbers alone, we will drop the conventional notation of naming it as Y-axis, and introduce a new one to make reference to this axis. From now on, we will call it the $Y^{\sim}$-axis (see Figure 1.1).

Definition 13. The X-axis is the **class of the real numbers** $\mathbb{R}$, and the $Y^{\sim}$-axis is the **class of the image real numbers** $\mathbb{R}^{\sim}$. The **origin of the coordinates** is the ordered pair $(0,0) = 0 = O$.

That is,

$$X = X\text{-axis} = \mathbb{R} \tag{1.3.9}$$

and

$$Y^{\sim} = Y^{\sim}\text{-axis} = \mathbb{R}^{\sim}. \tag{1.3.10}$$

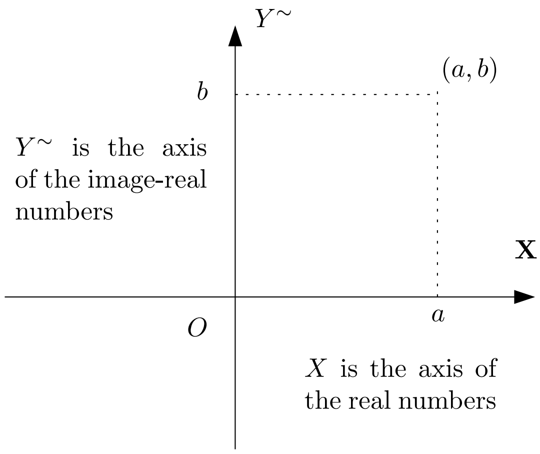

Figure 1.1: The coordinate plane of the ordered pairs

No number that belongs to the X-axis can concurrently belong to the Y$^{\sim}$, and vice versa; the exception is the zero, or origin O. This is expressed in set notation as:

$$X \cap Y^{\sim} = \big\{(0,0)\big\} = \{0\} = O \tag{1.3.11}$$

or

$$\mathbb{R} \cap \mathbb{R}^{\sim} = \big\{(0,0)\big\} = \{0\} = O. \tag{1.3.12}$$

Note that

$$O = \big\{(0,0)\big\} = \{0\}. \tag{1.3.13}$$

The set $\{0\}$ is an one-element set that means that the X- and $Y^{\sim}$-axes have nothing in common except the single member zero. If that shared element is removed, then nothing remains in common between them. Symbolically, this is expressed as

$$\big(\{X\} - \{0\}\big) \cap \big(\{Y^{\sim}\} - \{0\}\big) = \emptyset. \tag{1.3.14}$$

1.4 Addition and multiplication of ordered pairs

Definition 14. The operations of **addition**, +, and **multiplication**, *, for two ordered pairs (a, b) and (a', b') are defined as

$$(a, b) + (a', b') = (a + a', b + b') \tag{1.4.1}$$

$$(a, b) * (a', b') = (a * a', b * b'). \tag{1.4.2}$$

This definition, at the same time that does not violates the common addition and multiplication of real numbers, establishes an analogous definition for the image-real numbers.

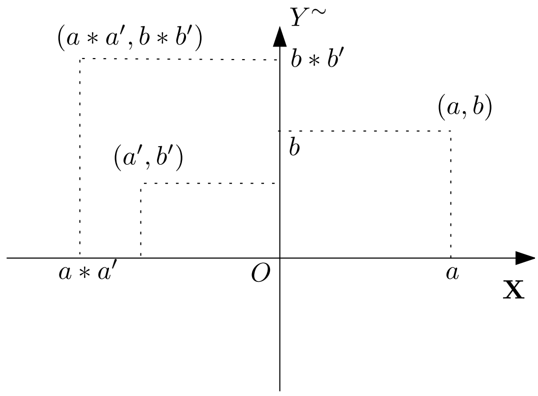

Figure 1.2: The multiplication of two ordered pairs

According to the above definition, a real number plus (or multiplied by) another real number is again a real number. Similarly, an image-real number plus (or multiplied by) another image-real number is again an image-real number (see Fig. 1.2). This can be seen below:

$$a + b = (a, 0) + (b, 0) = (a + b, 0) = a + b \tag{1.4.3}$$

and

$$a\tilde{} + b\tilde{} = (0, a) + (0, b) = (0, a + b) = (a + b)\tilde{}. \tag{1.4.4}$$

The result of adding a real number and an image-real number is immediate:

$$a + b\tilde{} = (a, 0) + (0, b) = (a + 0, 0 + b) = (a, b). \tag{1.4.5}$$

This addition cannot be carried further, implying that a real number plus an image-real number is not again a real number neither an image-real number. The addition of a real plus an image real is a non-axial ordered pair.

The converse is also true:

$$(a, b) = (a, 0) + (0, b) = a + b\tilde{}. \quad (1.4.6)$$

The interpretation of the above equation is that any ordered pair on the $XY\tilde{}$-plane can be decomposed as the addition of a real number plus an image-number.

The expression $a + b\tilde{}$ does not have any relation with the complex number expression $a+bi$. Their only common part is the real part, because for the complexs, the term bi, when squared, gives a negative real number:

$$(bi)^2 = -b^2 \quad (1.4.7)$$

while for the image-reals, when squared give:

$$(b\tilde{})^2 = (0, b) * (0, b) = (0, b^2) = (b^2)\tilde{}. \quad (1.4.8)$$

a positive non-real and non-imaginary number.

The fact that a positive image-number times another positive image-number is not a negative real, but another positive image-number again, should convince us that image real numbers are not the same as the imaginary numbers (which we will incorporate later).

Contrary to the above result, within the complex numbers, a positive imaginary times another positive imaginary gives as result a negative real.

On the interpretation of the X- and $Y\tilde{}$-axes that we have given, what this means is that the addition of two numbers on the X-axis is a number belonging to the X-axis again, and the addition of two numbers on the $Y\tilde{}$-axis is a number belonging to the $Y\tilde{}$-axis also.

However, when numbers belonging to different axes are added, the result is a non-axial ordered pair, that is, a point on the plane, but out of both axes.

Definition 15. In the ordered pair (a, b) we say that a is the **real part** of (a, b), and that $b\tilde{}$is the **image part** —not the imaginary part— of (a, b).

Respect to the multiplication of ordered pairs, a real number times another real number is again a real number, and an image-real times another image-real number is again an image-real number, as expected. This can be seen below:

$$a * b = (a, 0) * (b, 0) = (a * b, 0 * 0) = (ab, 0) = ab \quad (1.4.9)$$

and

$$ab\tilde{} * b\tilde{} = (0, a) * (0, b) = (0, a * b) = (0, ab) = (ab)\tilde{}. \quad (1.4.10)$$

Other special cases for the addition operations are:

$$(a,b)+(c,0)=(a+c,b+0)+(a+c,b) \tag{1.4.11}$$

$$(a,b)+(0,d)=(a+0,b+d)+(a,b+d). \tag{1.4.12}$$

The above two equations means that a non-axial ordered pair plus a real or image-real number results again in a non-axial ordered pair again.

For multiplication, one special case is:

$$(a,b)*(c,0)=(ac,0)=ac. \tag{1.4.13}$$

This implies that a non-axial ordered pair times a real number gives a real number as result.

Another special case is the following:

$$(a,b)*(0,d)=(0,bd)=(bd)\tilde{}. \tag{1.4.14}$$

This equation implies that a non-axial ordered pair times an image-real number gives an image-real number as result.

Factoring real and image-real numbers into nonreal or nonimage respectively is not defined.

Unexpected results come when we multiply (in any order) a real number by an image-real number:

$$a*(b\tilde{})=(a,0)*(0,b)=(0,0)=0 \tag{1.4.15}$$

and

$$a\tilde{}*(b)=(0,a)*(b,0)=(0,0)=0. \tag{1.4.16}$$

The fact that the product of ANY real number times ANY image number is ALWAYS zero will lead us to reexamine —in the definition of field— the assertion that a product of two numbers is zero if and only if at least one of its factors is zero, because for the two products examined above, this is not true.

The preceding result will lead to the following theorem:

THEOREM 1.1. *Let a and b be any two unknown numbers belonging to any of the X or $Y\tilde{}$-axes, then $ab=0$ if only if a belongs to X and b belongs to $Y\tilde{}$ or the converse. In symbols:*

$$ab=0 \tag{1.4.17}$$

if and only if

$$a\in X \qquad \textit{and} \qquad b\in Y\tilde{} \tag{1.4.18}$$

or

$$a \in Y\tilde{} \qquad \textit{and} \qquad b \in X \tag{1.4.19}$$

where

$$X = \mathbb{R} \qquad \textit{and} \qquad Y\tilde{} = \mathbb{R}\tilde{}. \tag{1.4.20}$$

Proof. (See the chapter on Theorem Proofs) □

Once the theorem is proven a corollary follows:

Corollary. If the product of two real numbers is zero, then at least one of them is zero

The corollary states the same condition (12) in the definition of field.

Similarly, for image numbers we can state that the product of two image-numbers is zero if at least one of them is zero.

Note then, that the fact that the product of two real numbers is zero if and only if one of them is zero has been reduced to a mere corollary of a one more general theorem. What the theorem simply states is that within the realm of the real and transreal numbers, in order that a product of two numbers be zero then at least one of then must be zero is no longer a necessary condition.

The Theorem 1.1 is one of the stepping stones of the *Foundations of the Transcomplex Number Theory.* The relevance of the theorem is that it states, when interpreted in relation with the X and $Y\tilde{}$-coordinates, that the product of any number of the X-axis times any other number of the $Y\tilde{}$-axis is always zero, no matter where they are (as long as they are on different axes).

Conversely, if the product of two non-zero numbers is zero, then each factor must be in a different axis.

This result may appear to be absurd and contradictory within the true spirit of the coordinates. But, it is not so, because to the effects of Theorem 1.1, the $Y\tilde{}$-axis is NOT a real numbers axis; only the X-axis is considered to be real.

In the geometrical representation, the X- and $Y\tilde{}$-axis are drawn at right angles to each other. Taking advantage of this fact and of Theorem 1.1 perpendicularity of numbers, and orthogonality of classes will be defined via products equal to zero.

1.4.1 Orthogonal classes

Definition 16. Two non zero ordered pairs are said to be mutually **perpendicular** if their product is zero, and two classes of ordered pairs are said to be mutually **orthogonal** if each element of one class is perpendicular to every element of the other class.

Note that perpendicularity refers to ordered pairs in particular while orthogonality refers to classes in general.

By this definition, and by Theorem 1.1, the real numbers class $\mathbb{R}$, that is, the X-axis, is orthogonal to the image numbers class $\mathbb{R}^{\sim}$, the $Y^{\sim}$-axis. Every "point" of one axis is perpendicular to every "point" in the other axis. That's the reason why both axes are orthogonal.

In simpler terms, that means that the X-axis is orthogonal ("perpendicular" in the common usage of the word) to the $Y^{\sim}$-axis because every real number of X is perpendicular to every image-real number of $Y^{\sim}$.

1.4.2 Scalar product for ordered pairs

If we have an ordered pair with each element having a common real number factor, say d, there is no way of taking that d outside of the pair, because

$$(da, db) = (d, d) * (a, b) \neq (d, 0) * (a, b) \tag{1.4.21}$$

but the real number d, the real factor, is not the same as the ordered pair (d, d)

$$d = (d, 0) \neq (d, d) \qquad \text{when} \qquad d \neq 0. \tag{1.4.22}$$

Thus, it is not correct to write $(da, db) = d * (a, b)$. In order to overcome that limitation, an additional definition is needed.

Definition 17. The **scalar product** of the real number d and the real ordered pair (a, b) is defined to be the ordered pair (da, db). In symbols:

$$d(a, b) = (da, db). \tag{1.4.23}$$

The scalar product and the *-product are different operations. Scalar product is defined for a real number times and ordered pair, while the *-product is defined for ordered pairs in general.

The scalar product does not uses the * symbol. Using it would imply the previous multiplication

$$d * (a, b) = (d, 0) * (a, b) = (da, 0) = da \tag{1.4.24}$$

while

$$d(a, b) = (da, db). \tag{1.4.25}$$

1.5 Other operations with ordered pairs

Definition 18. The **opposite** of the ordered pair (a, b) is the ordered pair $(-a, -b)$.

Definition 19. The **conjugate** of the ordered pair (a, b) is denoted by the symbol $\overline{(a, b)}$ and defined as:

$$\overline{(a, b)} = (a, -b). \tag{1.5.1}$$

Definition 20. The **inverse** of the non-axial real ordered pair (a, b), where $ab \neq 0$, is the real ordered pair (x, y) such that

$$(a, b) * (x, y) = (1, 1). \tag{1.5.2}$$

The inverse of (a, b) is denoted by $(a, b)^{-1}$.

Respect to the inverse of an ordered pair, realizing the multiplication stated in the above definition we obtain:

$$ax = 1 \qquad \text{and} \qquad by = 1. \tag{1.5.3}$$

Hence,

$$x = \frac{1}{a} = a^{-1} \qquad \text{and} \qquad y = \frac{1}{b} = b^{-1}. \tag{1.5.4}$$

Therefore

$$(a, b)^{-1} = (a^{-1}, b^{-1}). \tag{1.5.5}$$

If a or b are zero, this definition of inverse of ordered pair does not apply, because division by zero is not defined for reals. That implies that the real numbers and the image numbers do not have inverses (so far). In order to overcome that difficulty, a separate definition is needed for the reals and image reals.

Definition 21. The **inverse** of a real ordered pair is given by:

$$(a, 0)^{-1} = (a^{-1}, 0) \qquad \text{if} \qquad a \neq 0 \tag{1.5.6}$$

and

$$(0, b)^{-1} = (0, b^{-1}) \qquad \text{if} \qquad b \neq 0. \tag{1.5.7}$$

1.5.1 Division of ordered pairs

Definition 22. The **division** of the ordered pair (a, b) by the ordered (c, d) is realized as follows:

$$\frac{(a,b)}{(c,d)} = (a,b) * (c,d)^{-1}. \tag{1.5.8}$$

Or, in another terms:

$$\frac{(a,b)}{(c,d)} = (a * c^{-1}, b * d^{-}1). \tag{1.5.9}$$

This definition implies that real divided by real is real, and that image-real divided by image-real is also image-real as can be seen in the following two equalities. The first equality is for division of real numbers

$$\begin{aligned} \frac{a}{b} &= \frac{(a,0)}{(b,0)} \\ &= (a,0) * (b,0)^{-1} \\ &= (a,0) * (b^{-1},0) \\ &= (a * b^{-1}, 0) \\ &= a * b^{-1} \\ &= \frac{a}{b}. \end{aligned} \tag{1.5.10}$$

and the second equality if for division of image-real numbers

$$\begin{aligned} \frac{a^{\tilde{}}}{b^{\tilde{}}} &= \frac{(0,a)}{(0,b)} \\ &= (0,a) * (0,b)^{-1} \\ &= (0,a) * (0,b^{-1}) \\ &= (0, a * b^{-1}) \\ &= (a * b^{-1})^{\tilde{}} \\ &= (\frac{a}{b})^{\tilde{}}. \end{aligned} \tag{1.5.11}$$

But, a real number divided by an image-real number is zero as illustrated here dividing a real number a by an image-real number $b \neq 0$

$$\begin{aligned} \frac{a}{b\tilde{}} &= \frac{(a,0)}{(0,b)} \\ &= (a,0) * (0,b)^{-1} \\ &= (a,0) * (0,b^{-1}) \\ &= (0 * 0, 0 * b^{-1}) \\ &= (0,0) \\ &= 0. \end{aligned} \tag{1.5.12}$$

The same happens when an image-real number $a\tilde{}$ is divided by a real number $b \neq 0$:

$$\begin{aligned} \frac{a\tilde{}}{b} &= \frac{(0,a)}{(b,0)} \\ &= (0,a) * (b,0)^{-1} \\ &= (0,a) * (b^{-1},0) \\ &= (0 * b^{-1}, a * 0) \\ &= (0,0) \\ &= 0. \end{aligned} \tag{1.5.13}$$

The reason why division of any real by an image-real, and vice versa, is always zero, is because this definition is a mere consequence of the previous definition of multiplication of ordered pairs

1.5.2 Absolute value of ordered pairs

Definition 23. The **absolute value** of an ordered pair (a,b) is denoted by $|(a,b)|$ and defined by:

$$|(a,b)| = (|a|,|b|). \tag{1.5.14}$$

See Fig. 1.3.

It may seem strange that the absolute value of an ordered pair be defined as another ordered pair and not as a real number. The reason is that if we associate a pure real number with the absolute value, then not always will be true that the absolute value of the multiplication of two ordered pairs is equal to the multiplication of the absolute values. On the other hand, with the definition chosen, it is true that

$$|(a, b)| * |(a', b')| = |(a, b) * (a', b')| \tag{1.5.15}$$

because

$$|(a, b)| * |(a', b')| = (|a|, |b|) * (|a'|, |b'|) = (|aa'|, |bb'|). \tag{1.5.16}$$

At the same time,

$$|(a, b) * (a', b')| = |(aa', bb')| = (|aa'|, |bb'|). \tag{1.5.17}$$

Thus, the equality is established.

From this definition can be deduced the already stated absolute value for reals:

$$|(a, 0)| = (|a|, 0) = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases} \tag{1.5.18}$$

What happens when we try to find the absolute value of an image-real number? We get the following result:

$$|(0, b)| = (|0|, |b|) = (0, |b|) = |b|^{\sim}. \tag{1.5.19}$$

That is, for reals and image reals, the absolute value is simply

$$|(a, 0)| = |a| \tag{1.5.20}$$

and

$$|(0, b)| = |b|^{\sim}. \tag{1.5.21}$$

The absolute value of a real number is always a positive real number because it is used to measure the distance from a point in the X-axis to the origin of the coordinates. Thus, the fact that $|-a|$ and $|a|$ are the same is interpreted as that both, $-a$, and a, are at equal distance from the origin of coordinates.

On the other hand, the absolute value of an image-real number is another image-real number, although positive. That would imply that the notion of distance as a unique real number is lost. However, looking closely, what it really means is that the absolute value of a real number is a positive real number because the distance is measured along the X-axis, and the absolute value of an image-real number is another positive image-real number because that distance is measured along the $Y^{\sim}$-axis.

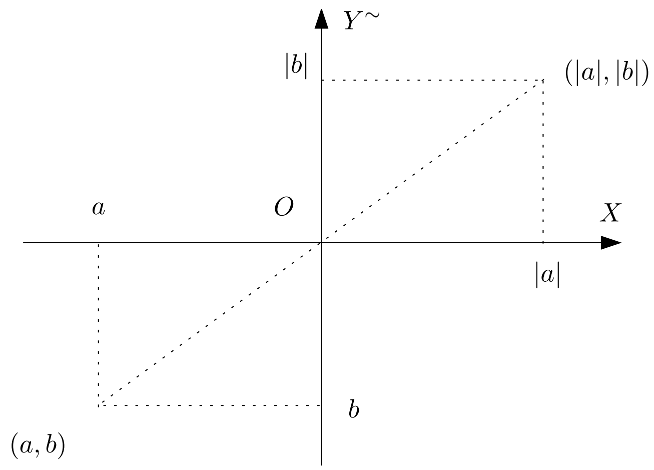

Figure 1.3: The absolute value of an ordered pair is another ordered pair

We'll return again to the notion of absolute value later when dealing with complex numbers and when dealing with transcomplex numbers in general.

1.5.3 The norm of ordered pairs

Definition 24. The **norm**of an ordered pair (a, b), denoted by $\|(a, b)\|$ is defined by

$$\|(a,b)\| = |\sqrt{(a^2, b^2)}|. \tag{1.5.22}$$

This definition implies that the norm of an ordered pair is always a positive real number. Since the common usage is to accept the concept of absolute value as a unique real number, with this definition we arrive at that notion.

For reals and image-reals, the norm is reduced to:

$$\|(a,0)\| = \|a\| = |a| \tag{1.5.23}$$

because

$$\begin{aligned}\|(a,0)\| &= |\sqrt{(a^2, 0^2)}| \\ &= |\sqrt{a^2}| \\ &= |a|\end{aligned} \tag{1.5.24}$$

and

$$\|(0, b)\| = \|b^{\sim}\| = |b| \tag{1.5.25}$$

because

$$\begin{aligned} \|(0, b)\| &= |\sqrt{(0^2, b^2)}| \\ &= |\sqrt{b^2}| \\ &= |b|. \end{aligned} \tag{1.5.26}$$

Making a comparison between absolute values of real numbers and image-real numbers we can see that they are unequal, except when $a = 0$:

$$|a| \neq |a^{\sim}| \tag{1.5.27}$$

but, for the norm of a real number and image-real number we can see that they are always equal:

$$\|a\| = \|a^{\sim}\|. \tag{1.5.28}$$

Note that

$$|a| * |a^{\sim}| = 0 \tag{1.5.29}$$

but, on the contrary,

$$\|a\| * \|a^{\sim}\| \neq 0 \qquad \text{if} \qquad a \neq 0. \tag{1.5.30}$$

The implications of the previous results are that $|a|$ and $|a^{\sim}|$ are perpendicular numbers because their product is zero. This is because absolute values are positive, but not necessarily reals. The numbers $\|a\|$ and $\|a^{\sim}\|$ are not perpendicular numbers because both exist on the positive part of the X-axis. This is because norms are always positive and reals.

The norm of an ordered pair expresses the distance from the ordered pair to the origin of coordinates (see Fig. 1.4).

Here we are using the concept of "distance" in the sense of the physical spread, or extent between extremes.

Ordered pairs with equal norm make a circumference around the origin of coordinates.

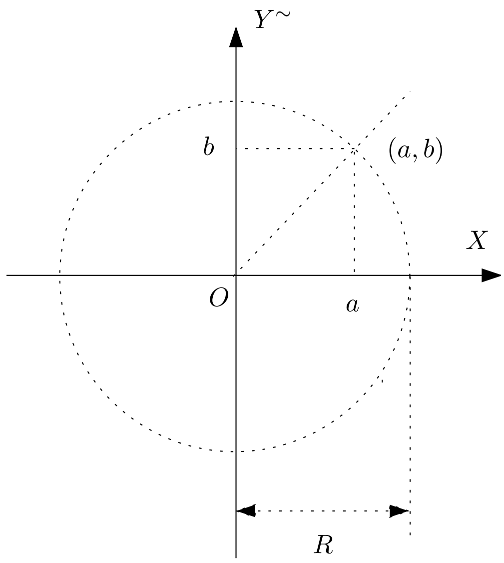

Figure 1.4: The norm of an ordered pair is the real number **R**

1.5.4 The argument of ordered pairs

Definition 25. The **principal argument** of a real ordered pair (a, b), where $a \neq 0$, is denoted by $Arg(a, b)$ and is defined by

$$Arg(a, b) = \tan^{-1} \frac{b}{a}, \qquad \text{where} \qquad a \neq 0. \tag{1.5.31}$$

Since the tangent function is cyclic, there are many other arguments for any ordered pair. For an ordered pair (a, b), the other arguments are represented by the lower letter symbol **arg(a, b)** and defined as:

$$arg(a, b) = Arg(a, b) + 2k\pi \qquad \text{for} \qquad (k = 1, 2, 3, \ldots). \tag{1.5.32}$$

The argument of an ordered pair can be interpreted as the angle q that a ray departing from the origin to the point itself makes in relation with the positive side of the X-axis (see Fig. 1.5). A positive angle is measured counterclockwise. Points, that is, ordered pairs, with equal argument coincide with a radius vector departing from the origin of coordinates.

1.6 The ordered pairs field

1.6.1 The ordered pairs field theorem

The following two laws are part of the ordered pairs field.

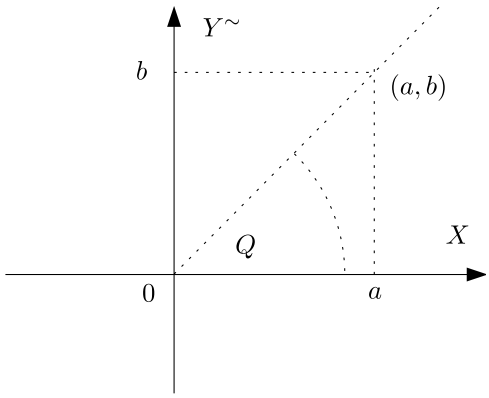

Figure 1.5: The argument of an ordered pair is the angle **Q**

Definition 26. Exponentiation

$$(a, b)^n = (a^n, b^n). \tag{1.6.1}$$

Radicalization In particular:

$$\sqrt{(a, b)} = (\sqrt{a}, \sqrt{b}). \tag{1.6.2}$$

THEOREM 1.2. *The class $\mathbb{O}$, that is, the class of all real numbers ordered pairs, together with the operations + and *, make a field.*

Proof. (See the chapter on Theorem Proofs) □

1.6.2 Exponentiation and radicalization of image-real numbers

Definition 27. The rules for **exponentiation** and **radicalization** of image-real numbers are:

$$(a^{\sim})^0 = 1^{\sim} \tag{1.6.3}$$

$$(a^{\sim})^1 = a^{\sim} \tag{1.6.4}$$

$$(a^{\sim})^{n+1} = (a^{\sim})^n * (a^{\sim}) \tag{1.6.5}$$

$$(a^{\sim})^m * (a^{\sim})^n = (a^{\sim})^{m+n} \tag{1.6.6}$$

$$((a^{\sim})^m)^n = (a^{\sim})^{m*n} \tag{1.6.7}$$

$$(a^{\sim})^{-n} = \frac{1}{(a^{\sim})^n} \tag{1.6.8}$$

$$\frac{(a^{\sim})^m}{(a^{\sim})^n} = (a^{\sim})^{m-n} \qquad \text{if} \qquad a \neq 0 \tag{1.6.9}$$

$$\left(\frac{a^{\sim}}{b^{\sim}}\right)^n = \frac{(a^{\sim})^n}{(b^{\sim})^n} \qquad \text{if} \qquad b \neq 0 \tag{1.6.10}$$

$$\sqrt[n]{a^{\sim}} = r^{\sim} \qquad \text{if} \qquad (r^{\sim})^n = a^{\sim} \tag{1.6.11}$$

$$(a^{\sim})^{\frac{1}{n}} = \sqrt[n]{a^{\sim}} \qquad \text{if} \qquad n \neq 0 \tag{1.6.12}$$

$$(a^{\sim})^{\frac{m}{n}} = \left(\sqrt[n]{a^{\sim}}\right)^m \qquad \text{if} \qquad n \neq 0. \tag{1.6.13}$$

Chapter 2

Complex Numbers

2.1 Introduction

The complex numbers are the *sine qua non* of the transcomplex numbers. No mayor changes are made to the classic definition of them and the operations of addition and multiplication are identical with those found in complex variables and calculus textbooks. However, parallel to the previously defined image-numbers, the concept of imaginary number is presented as a natural implication of the concept of ordered pair.

The definition of absolute value for a complex number is slightly different to the current usage. Here is found under the new term of norm. None of the changes has been done arbitrarily, but always keeping in mind the future usage under the transcomplex numbers concept. The usual trigonometric representation of the complexs is also covered together with a theorem that makes relevant the additive properties of arguments under multiplication of transcomplexs.

The development taken here of the complex numbers is similar to that seen in current textbooks, with the exception of a minor change in notation.

2.2 Imaginary and image-imaginary numbers

We know that the multiplication of two ordered pairs is as follows:

$$(a, b) * (a', b') = (aa', bb') \tag{2.2.1}$$

but, when do we have the following case?

$$(a, b) * (a', b') = (-aa', -bb'). \tag{2.2.2}$$

The answer to that question is that we'll never be confronted with this case, because it clearly represents a contradiction in signs.

However, it may be possible that under a new kind of numbers, numbers that we temporarily add the question mark (?) to it like $a?$ and $b?$, we could have:

$$\begin{aligned}(a?, b?) * (a'?, b'?) &= (aa'?, bb'?) \\ &= (-aa', -bb').\end{aligned} \tag{2.2.3}$$

That means that we need to introduce a new kind of number that could produce such a result. In order to produce that particular result with multiplication, the numbers $a?$ and $b?$ must be of such a nature that

$$a * a' = -aa' \tag{2.2.4}$$

and

$$b * b' = -bb'. \tag{2.2.5}$$

Since a set can be chosen to be made of any kind of objects, then an ordered pair need not necessarily be made of real numbers only. In view of that, we can introduce that new kind of ordered pair.

That kind of number is called imaginary numbers and they are formally defined as follows substituting the question mark by the symbol i.

Definition 28. An **imaginary number** ai is the ordered pair $(ai, 0)$ such that

$$(ai, 0) * (ai, 0) = (-a^2, 0) \tag{2.2.6}$$

and an **image-imaginary number** ai is the ordered pair $(0, ai)$ such that

$$(0, ai) * (0, ai) = (0, -a^2). \tag{2.2.7}$$

Of special interest among the imaginary numbers is the imaginary units defined below.

The notation can be abbreviated as

$$ai = (ai, 0) \qquad \text{and} \qquad ai\tilde{} = (0, ai\tilde{}). \tag{2.2.8}$$

The image-imaginary number $ai\tilde{}$ can also be called a **transimaginary number**.

The **zero imaginary number**, $(0, 0)$ is at the same time the **zero image-imaginary number**. That is:

$$(0, 0) = (0, 0)\tilde{} = 0. \tag{2.2.9}$$

Definition 29. The **unit imaginary number** is defined to be the ordered pair $(i, 0)$ such that

$$(i, 0) * (i, 0) = (-1, 0) \tag{2.2.10}$$

and the **unit image-imaginary number**, or the **transimaginary unit**, is defined to be the ordered pair $(0, i)$ such that

$$(0, i) * (0, i) = (0, -1). \tag{2.2.11}$$

The ordered pair $(i, 0)$ can be abbreviated as i, and the ordered pair $(0, i)$ as $i\tilde{}$. Simply written,

$$i * i = -1 \qquad \text{and} \quad i\tilde{} * i\tilde{} = -1\tilde{}. \tag{2.2.12}$$

The symbol $\mathbb{I}$ will denote the **class of all imaginary numbers**, and the symbol $\mathbb{I}\tilde{}$ will denote the **class of all image-imaginary numbers**. At the same time we'll call the $\mathbb{I}$-class the **imaginary numbers axis**, or simply the **imaginary axis** and denote it by the I-axis. In the same way, the $\mathbb{I}\tilde{}$ class will be called the **image-imaginary numbers axis** or simply the **image-imaginary axis**.

The imaginary axis is better known as the iZ-axis, and we'll keep that notation also. Parallel to this, $i\tilde{}Z$ will also denote the image-imaginary axis.

Using the previously stated definition of what an ordered pair is, we have that an imaginary number ai is the same as:

$$(ai, 0) = \big\{\{ai\}, \{(ai, 0)\}\big\} \tag{2.2.13}$$

and

$$(0, ai) = \big\{\{0\}, \{(0, ai)\}\big\}. \tag{2.2.14}$$

That letter i is not an arbitrary letter, but it represents a number with the property that $i * i = -1$.

We cannot assert that the ordered pair $(i, 0)$ belongs to the real numbers class $\mathbb{R}$ just because the second entry of $(i, 0)$ is zero. Real numbers have their second entry equal to zero, making $(i, 0)$ a candidate to be real, but $(i, 0)$ cannot belong to the reals class $\mathbb{R}$ because no real number, positive or negative, times itself is negative. This fact by itself leaves i outside of the class $\mathbb{R}$.

On the other hand, the image-real numbers are of the form $(0, b)$, and when multiply any image-real number by itself what we obtain is:

$$(0, b) * (0, b) = (0, b^2) \tag{2.2.15}$$

but b^2 is always positive, no matter the sign of b, hence an image-real number is never the same as an imaginary number.

Using the set-theory notation, we have that for every imaginary number ai:

$$ai \in \mathbb{I} \tag{2.2.16}$$

but

$$ai \notin \mathbb{R} \qquad \text{and} \qquad ai \notin \mathbb{R}\tilde{} \tag{2.2.17}$$

and for every image-imaginary number $ai\tilde{}$:

$$ai\tilde{} \in \mathbb{I}\tilde{} \tag{2.2.18}$$

and

$$ai\tilde{} \notin \mathbb{R}, \qquad ai\tilde{} \notin \mathbb{R}\tilde{} \qquad \text{and} \qquad ai \notin \mathbb{I}. \tag{2.2.19}$$

2.3 Imaginary ordered pairs

Definition 30. When we choose an ordered pair with both entries to be imaginary numbers, say ai and bi, then we say that (ai, bi) is an **imaginary numbers ordered pair**, or an **ordered pair of imaginary numbers**. Associated with the ordered pair (ai, bi) is the ordered pair (bi, ai) which is called the **image** of the ordered pair (ai, bi), or simply, the **image-imaginary ordered pair** (bi, ai). A tilde ($\tilde{}$) above the symbol denotes the concept of image. Thus,

$$(ai, bi)\tilde{} = (bi, ai). \tag{2.3.1}$$

Note that from the above definition it follows that

$$\big((ai, bi)\tilde{}\big)\tilde{} = (bi, ai)\tilde{} = (ai, bi). \tag{2.3.2}$$

As previously used for real ordered pairs, imaginary ordered pairs with none of its components being equal to zero will be called **non-axial imaginary ordered pairs**.

2.3.1 The imaginary coordinate plane

Intuitively, we think of an imaginary plane as made by the intersection of the I and $I\tilde{}$-axes. The point of intersection of all four axes is the ordered pair $(0, 0) = 0 = O$ which coincidentally is the same point of intersection of the X and $Y\tilde{}$ axes

That common origin is important to us because we have being constantly tied to the principle (axiom) of uniqueness of numbers. That implies to us that there cannot be separate origins of coordinates if that origin is a unique number. Therefore, the $XY\tilde{}$plane and the $II\tilde{}$plane must be perpendicular planes and that poses us a visualization problem since with the exception of the origin nothing else is coincidental among those two planes.

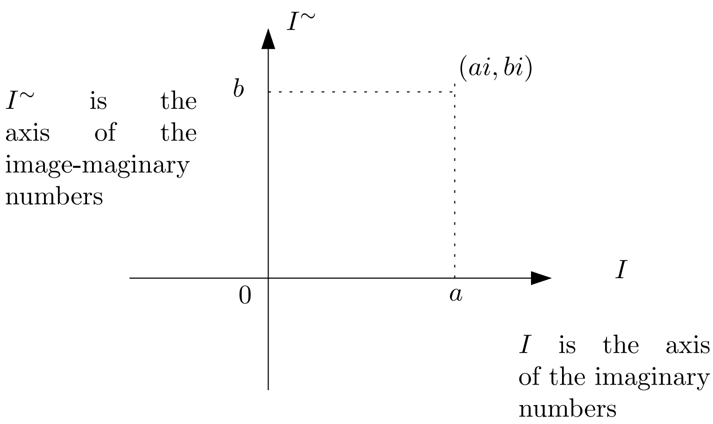

Figure 2.1: The coordinate plane of the imaginary ordered pairs

No number that belongs to the I-axis can concurrently belong to the $I\tilde{}$-axis, and vice versa; the exception is the zero, or origin O. This is expressed in set notation as:

$$I \cap I\tilde{} = \big\{(0, 0)\big\} = \{0\} = 0 \tag{2.3.3}$$

or

$$\mathbb{I} \cap \mathbb{I}\tilde{} = \big\{(0, 0)\big\} = \{0\} = 0. \tag{2.3.4}$$

Note that

$$O = \big\{(0, 0)\big\} = \{0\}. \tag{2.3.5}$$

The set $\{0\}$ is a one-element set; that means that the I- and $I\tilde{}$-axes have nothing in common except the single member zero. If that shared element is removed, then nothing remains in common between the I- and the $I^{\sim}$-axes. Symbolically, this is expressed as

$$\big(\{I\} - \{0\}\big) \cap \big(\{I^{\sim}\} - \{0\}\big) = \emptyset. \tag{2.3.6}$$

There are many "imaginary" planes. The plane $II^{\sim}$is just one of them. We mention this because here we are working with the imaginary ordered pairs only. Later, in Chapter 4, devoted to the coordinate system S^4, we'll see more of them.

2.4 Operations with imaginary ordered pairs

Definition 31. The operations of **addition**, +, and **multiplication**, *, for two imaginary ordered pairs (ai, bi) and $(a'i, b'i)$ are defined as

$$(ai, bi) + (a'i, b'i) = (ai + a'i, bi + b'i) \tag{2.4.1}$$

and

$$(ai, bi) * (a'i, b'i) = (ai * a'i, bi * b'i). \tag{2.4.2}$$

2.4.1 Addition of imaginary ordered pairs

Respect to the addition of imaginary ordered pairs, the definition implies that an imaginary number plus another imaginary number is again an imaginary number. Similarly, an image-imaginary number plus another image-imaginary number is again an image-imaginary number. This can be seen below:

$$\begin{aligned} ai + bi &= (ai, 0) + (bi, 0) \\ &= (ai + bi, 0 + 0) \\ &= ((a + b)i, 0) \end{aligned} \tag{2.4.3}$$

and

$$\begin{aligned} ai^{\sim} + bi^{\sim} &= (0, ai) + (0, bi) \\ &= (0 + 0, ai + bi) \\ &= (0, (a + b)i). \end{aligned} \tag{2.4.4}$$

The result of adding an imaginary number and an image-imaginary number is immediate:

$$\begin{aligned} ai + bi\tilde{} &= (ai, 0) + (0, bi) \\ &= (ai + 0, 0 + bi) \\ &= (ai, bi). \end{aligned} \tag{2.4.5}$$

This addition cannot be carried further, implying that an imaginary number plus an image-imaginary number is not again an imaginary number neither an image-imaginary number. The addition of an imaginary plus an image-imaginary number is a non-axial imaginary ordered pair.

The converse is also true:

$$(ai, bi) = (ai, 0) + (0, bi) = ai + bi\tilde{}. \tag{2.4.6}$$

The interpretation of the above equation is that any ordered pair on the $II\tilde{}$-plane can be decomposed as the addition of an imaginary number plus an image-imaginary number.

The expression $ai + bi\tilde{}$ does not have any relation with the complex number expression $a + bi$.

On the interpretation of the I- and $I\tilde{}$-axes that we have given, what this means is that the addition of two numbers on the I-axis is a number belonging to the I-axis again, and the addition of two numbers on the $I\tilde{}$-axis is a number belonging to the $I\tilde{}$-axis.

However, when imaginary numbers belonging to different axes are added, the result is a non-axial ordered pair, that is, a point on the imaginary plane, but out of both axes.

2.4.2 Multiplication of imaginary ordered pairs

Note that when the multiplication is expanded, it is obtained:

$$\begin{aligned} (ai, bi) * (a'i, b'i) &= (ai * a'i, bi * b'i) \\ &= (a * a' * i * i, b * b' * i * i) \\ &= (a * a' * -1, b * b' * -1) \\ &= (-aa', -bb'). \end{aligned} \tag{2.4.7}$$

On the other hand, an imaginary number times another imaginary number is a real number. Similarly, an image-imaginary number times another image-imaginary number is

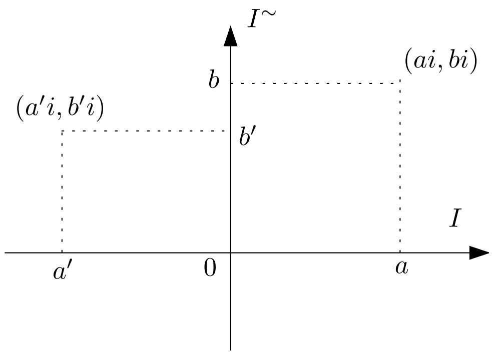

Figure 2.2: The multiplication of two imaginary ordered pairs is not again an imaginary ordered pair

again an image-real number:

$$\begin{aligned} ai * bi &= (ai, 0) * (bi, 0) \\ &= (ai * bi, 0 * 0) \\ &= (a * b * -1, 0) \\ &= (-ab, 0) \end{aligned} \tag{2.4.8}$$

and

$$\begin{aligned} ai^{\sim} * bi^{\sim} &= (0, ai) * (0, bi) \\ &= (0 * 0, ai * bi) \\ &= (0, a * b * -1) \\ &= (0, -ab). \end{aligned} \tag{2.4.9}$$

2.4.3 Scalar product of imaginary ordered pairs

Sometimes we need to factor-out an imaginary number $(ai, 0)$ to decompose it using the imaginary unit concept. To do it, the scalar product is needed, however, the scalar product, defined previously is enough for our purposes, but it is reintroduced because now we are dealing with another type of number.

Definition 32. The **scalar product** of the real number c and the imaginary ordered pair (ai, bi) is performed as follows:

$$c(ai, bi) = (cai, cbi). \tag{2.4.10}$$

It follows then that the scalar product of a number c and the imaginary number ai is

$$c(ai, 0) = (cai, 0) = cai. \tag{2.4.11}$$

Similarly, for the scalar product of the image imaginary number $ai\tilde{}$:

$$c(0, ai) = (0, cai) = cai\tilde{}. \tag{2.4.12}$$

The scalar product is defined for a real number c times an imaginary ordered pair, but this definition does not allows us to use the unit imaginary number i to be taken out of the ordered pair since i is not a real number. To overcome this we also define the imaginary scalar product as $(ai, bi) = (a, b)i$.

Definition 33. The **imaginary scalar product** of the real numbers ordered pair (a, b) and the unit imaginary number i is defined to be:

$$(a, b)i = (ai, bi). \tag{2.4.13}$$

2.4.4 Absolute value of an imaginary ordered pair

Definition 34. The **absolute value** of the imaginary ordered pair (ai, bi) is defined to be:

$$\big|(ai, bi)\big| = \big|(a, b)\big|i. \tag{2.4.14}$$

That means that the absolute value of an imaginary ordered pair is another imaginary ordered pair.

By the definition of absolute value of ordered pairs introduced in the previous chapter we have that

$$\big|(a, b)\big| = \big(|a|, |b|\big) \tag{2.4.15}$$

hence,

$$\big|(ai, bi)\big| = \big(|a|, |b|\big)i. \tag{2.4.16}$$

2.4.5 Multiplication of imaginary numbers and image-imaginary numbers

Let ai be an imaginary number,

$$ai = a(i, 0) = (ai, 0) \tag{2.4.17}$$

and let $bi\tilde{}$ be an image-imaginary number,

$$bi\tilde{} = b(0, i) = (0, bi). \tag{2.4.18}$$

By definition, the multiplication of ai and $bi\tilde{}$ is carried as follows:

$$\begin{aligned} ai * bi\tilde{} &= a(i, 0) * b(0, i) \\ &= ab(i * 0, 0 * i) \\ &= ab(0, 0) \\ &= 0. \end{aligned} \tag{2.4.19}$$

So, an imaginary number multiplied by an image-imaginary number results in a product of zero. Expressed in set-theory notation:

$$\forall a \in \mathbb{I} \wedge \forall b\tilde{} \in \mathbb{I}\tilde{} \Rightarrow a * b\tilde{} = 0. \tag{2.4.20}$$

2.4.6 Orthogonality of the I and $I^{\sim}$-axes

According to the definition of orthogonality previously stated in Chapter 1, that means that the I-axis and the $I\tilde{}$-axis are also orthogonal axes because the product of an imaginary number belonging to the I-axis times an image-imaginary number belonging to the $I\tilde{}$-axis results always in zero.

However, we need that all four axes be mutually orthogonal. We will not venture now into defining when any two axes of the four already defined are orthogonal, and we'll wait until the transcomplex numbers are introduced to make the final and definitive definition of orthogonality.

2.5 Complex numbers

Definition 35. A **complex number** or **complex ordered pair** is an ordered pair composed of a real number and an imaginary number. The notation $C = a + ci$ or $C = (a, ci)$ will be used indistinguishably to denote a complex number. We will denote by $\mathbb{C}$ the **class of all complex numbers**.

Since a complex number is an ordered pair, then recurring to the set-notation, a complex number $C = (a, ci)$ is:

$$(a, ci) = \big\{\{a\}, \{a, ci\}\big\}. \tag{2.5.1}$$

The equality relation for complex numbers does not need another definition because it can be deduced by the equality relation of sets. Therefore, two complex numbers $C = (a, ci)$ and $C' = (a', c'i)$ are equal if and only if $a = a'$ and $c = c'$.

Another way to describe the equality of complex numbers in terms of real ordered pairs is:

$$(a, ci) = (a', c'i) \tag{2.5.2}$$

if and only if

$$(a, c) = (a', c'). \tag{2.5.3}$$

Definition 36. In a complex number $C = a + ci$, a is called the **real part** of C, and this is denoted by $Re(C)$. Similarly, ci is called the **imaginary part** of C and denoted by the symbol $Im(C)$. In symbols

$$Re(C) = Re(a, ci) = a \tag{2.5.4}$$

$$Im(C) = Im(a, ci) = ci. \tag{2.5.5}$$

Within the complex numbers realm, the real numbers are obtained when we choose the imaginary part to be zero. Conversely, imaginaries are complexs with the real part equal to zero. In symbols:

$$(a, 0i) \in \mathbb{R} \qquad \text{and} \qquad (0, ci) \in \mathbb{I}. \tag{2.5.6}$$

It can be effortless proved that

$$\mathbb{R} \subset \mathbb{C} \qquad \text{and} \qquad \mathbb{I} \subset \mathbb{C}. \tag{2.5.7}$$

2.5.1 The complex numbers plane

Since a complex number is made of a real number and an imaginary number, the plotting of complexs (and complex variables in general) must be on a plane made by the intersection of the $\mathbb{R}$ and the $\mathbb{I}$-axes.

Thus, the plotting of complex numbers does not require the introduction of any axis apart from those that we have already defined.

The only question remaining is the orthogonality of the $\mathbb{R}$ and $\mathbb{I}$-axes so we can rightfully plot complex numbers, but, as mentioned above, we'll wait until the introduction of the transcomplex numbers to state its orthogonality.

Meanwhile, the plotting of complex numbers follows the same rules as when plotting real ordered pairs; i.e., on the real numbers axis $\mathbb{R}$ we plot the real part of the complex numbers, and on the imaginary numbers axis $\mathbb{I}$ we plot the imaginary part.

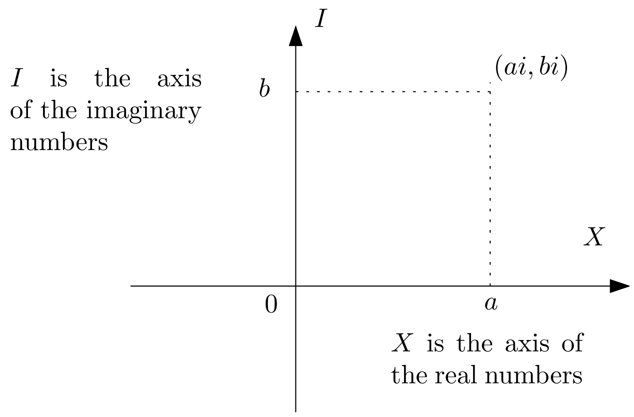

Figure 2.3: The Coordinate plane of the complex numbers

2.5.2 Operations with complex numbers

After the definition of complex number, the next step is to define the permissible operations with them. The definitions for the operations of addition, +, and multiplication, *, already stated for real and imaginary numbers cannot be immediately extended to the complex numbers because they are neither real numbers nor imaginary numbers. Thus, different operations must be defined.

Definition 37. The operations of **addition**, +, and **multiplication**, *, for two complex numbers $C = (a, ci)$ and $C' = (a', c'i)$ are defined as follows:

$$\begin{aligned} C + C' &= (a + ci) + (a', c'i) \\ &= (a + a', c + c') \end{aligned} \tag{2.5.8}$$

and

$$\begin{aligned} C * C' &= (a, ci) * (a', c'i) \\ &= (a * a' + ci * c'i, a * c'i + a' * ci). \end{aligned} \tag{2.5.9}$$

Note that this multiplication can be further simplified as follows:

$$\begin{aligned} C * C' &= (a, ci) * (a', c'i) \\ &= (a * a' + ci * c'i, a * c'i + a' * ci) \\ &= (aa' + cic'i, ac'i + a'ci) \\ &= (aa' - cc', ac'i + a'ci) \\ &= \big(aa' - cc', (ac' + a'c)i\big). \end{aligned} \tag{2.5.10}$$

The operation of addition of complex numbers remained similar to the addition of real ordered pairs and addition of imaginary ordered pairs, while the operation of multiplication is totally different for the complexs in comparison with the multiplication of real numbers ordered pairs and multiplication of imaginary ordered pairs.

2.5.3 Addition of complex numbers

Regarding to the addition of complex numbers, when their imaginary part is zero, we have that —as expected— real plus real is real:

$$(a, 0i) + (a', 0i) = (a + a', 0i) = a + a'. \tag{2.5.11}$$

Analogously, imaginary plus imaginary is again imaginary:

$$\begin{aligned} (0, ci) + (0, c'i) &= (0, ci + c'i) \\ &= (0, (c + c')i) \\ &= (c + c')i. \end{aligned} \tag{2.5.12}$$

2.5.4 Multiplication of complex numbers

In multiplication, real times real is real because for two complexs $C = (a, ic)$ and $C' = (a', ic')$ with imaginary parts c and c' equal zero in both numbers we have:

$$\begin{aligned}
C * C' &= (a, 0i) * (a', 0i) \\
&= (a * a' + 0i * 0i, a * 0i + a' * 0i) \\
&= \big(a * a' + (0 + 0)i, (a * 0)i + (0 * a')i\big) \\
&= (aa' + 0, 0i + 0i) \\
&= (aa', 0) \\
&= aa'. \qquad (2.5.13)
\end{aligned}$$

When $C = (a, ci)$, where $a = 0$, and $C' = (a', c'i)$, where $a' = 0$ we are having two imaginary numbers: $C = (0, ci)$ and $C' = (0, c'i)$. Their product also results in a real number as can be seen below:

$$\begin{aligned}
C * C' &= (0, ci) * (0, c'i) \\
&= (0 * 0 + ci * c'i, 0 * c'i + 0 * ci) \\
&= \big(0 + (c * c') * (i * i)), 0 + 0\big) \\
&= \big(- (c * c'), 0\big) \\
&= -cc'. \qquad (2.5.14)
\end{aligned}$$

On the other hand, a real number multiplied by an imaginary number is another imaginary number:

$$\begin{aligned}
C * C' &= (a, 0i) * (0, c'i) \\
&= (a * 0 + 0i * c'i, a * c'i + 0 * ci) \\
&= \big(0 - 0, (ac' + 0 * a')i\big) \\
&= \big(0, (ac' + 0)i\big) \\
&= (0, ac'i) \\
&= ac'i. \qquad (2.5.15)
\end{aligned}$$

Let $C = (a, 0)$ and $C' = (a', c')$ be a real number and a complex number respectively. Then, their product is:

$$\begin{aligned}
C * C' &= (aa' + 0, ac'i + 0) \\
&= (aa', ac'i). \qquad (2.5.16)
\end{aligned}$$

From the above result we deduce that for a real number a and another complex number $C' = (a', c'i)$:

$$a * C' = (aa', ac'). \tag{2.5.17}$$

Hence, for complex numbers the scalar product needs no new definition.

THEOREM 2.1. *The unit element under multiplication of the complex numbers field is the complex number $1 + 0i = 1$. That is, for every complex number C:*

$$C * 1 = C. \tag{2.5.18}$$

Proof. (See the chapter on Theorem Proofs) □

THEOREM 2.2. *The inverse of the complex number $C = a + ci = (a, ci)$ is the complex number C^{-1} given by:*

$$\begin{aligned} C^{-1} &= \frac{a}{a^2 + c^2} - \frac{c}{a^2 + c^2} \\ &= \left(\frac{a}{a^2 + c^2}, -\frac{c}{a^2 + c^2}\right). \end{aligned} \tag{2.5.19}$$

Proof. (See the chapter on Theorem Proofs) □

2.5.5 The absolute value and the norm of a complex number

Definition 38. The **absolute value** of a complex number $C = a + ci$ is denoted by $|C|$, and defined by:

$$|C| = |a + ci| = |a| + |c|i. \tag{2.5.20}$$

For the reader familiar with the classic definition of absolute value of a complex number, where the absolute value of a complex numbers is defined as a real number, may sound very strange that here the absolute value of a complex number is defined as another complex number and not as a real number.

However, it is under the concept of norm —introduced next— that the reader will find that classic definition.

Definition 39. The **norm** of a complex number $C = a+ci$ is denoted by $\| C \|$, and defined by:

$$\|C\| = \|a + ci\| = |\sqrt{a^2 + c^2}|. \tag{2.5.21}$$

The reader should take notice that the symbol $\| \ \|$ does not mean a double absolute value since this is nonsense due to the fact that being an absolute value a positive real number, taking another absolute value will yield no new results. The symbol $\| \ \|$ should be taken as a symbol for a new concept.

There is a distinction between the absolute value of a complex number an its norm. The absolute value is another complex number, while the norm is a single real number. What we are calling here the norm, is what customarily is called the absolute value, and what we are calling here the absolute value does not have analogy in the common usage of complex numbers.

Complex numbers with equal norm lie on a circumference with center at the origin of coordinates, and radius equal to the norm. However, complex numbers with equal absolute value lie at the corners of a rectangle.

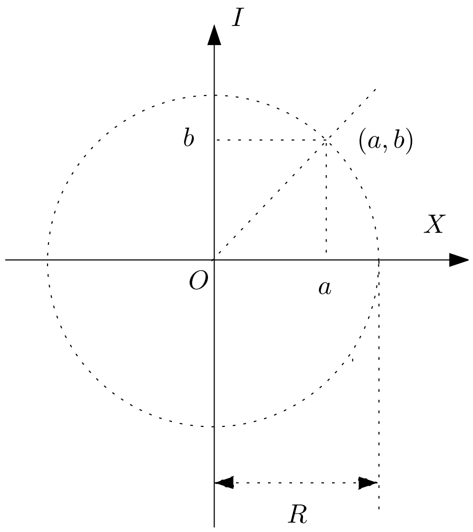

Figure 2.4: The norm of a complex number

THEOREM 2.3. *Let $C = a + ci$ and $C' = a' + c'i$ be any two complex numbers, then*

$$|C| * |C'| \neq |C * C'| \tag{2.5.22}$$

but on the contrary,

$$\|C\| * \|C'\| = \|C * C'\|. \tag{2.5.23}$$

Proof. (See the chapter on Theorem Proofs) □

2.5.6 The argument of a complex number

Definition 40. The **principal argument** of a complex number $C = a + ci$ is denoted by $Arg(C)$, and is defined by:

$$Arg(C) = \tan^{-1} \frac{c}{a}, \qquad \text{if} \qquad a \neq 0. \tag{2.5.24}$$

Since the tangent function is cyclic, there are many other arguments for any complex number C. The other arguments are represented by the lower letters symbol $arg(C)$. Therefore,

$$arg(C) = Arg(C) + 2k\pi \qquad \text{for} \qquad (k = 1, 2, 3, \ldots). \tag{2.5.25}$$

The principal argument is defined for

$$-\pi < Arg(C) \leq \pi. \tag{2.5.26}$$

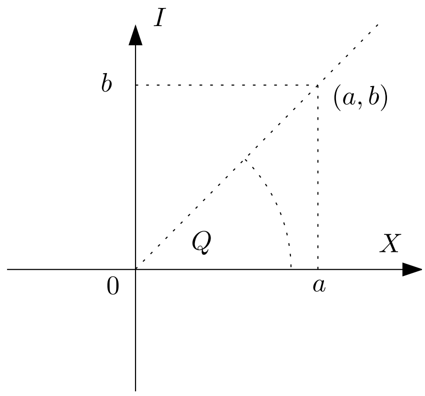

Figure 2.5: The argument of a complex number

THEOREM 2.4. *Let $C = a + ci$, then*

$$\|C^{-1}\| = \|C\|^{-1} \tag{2.5.27}$$

and

$$Arg(C^{-1}) = -Arg(C). \tag{2.5.28}$$

Proof. (See the chapter on Theorem Proofs) □

THEOREM 2.5. *Let $C = a + ci$ and $C' = a' + c'i$ be any two complex numbers. Then:*

$$Arg(C * C') = Arg(C) + Arg(C') \tag{2.5.29}$$

and

$$Arg\left(\frac{C}{C'}\right) = Arg(C) - Arg(C'). \tag{2.5.30}$$

Proof. (See the chapter on Theorem Proofs)

□

2.5.7 The trigonometric form of a complex number

In a complex number $C = a + ci$, we can take any real number r as factor of C provided the corresponding division of a and c by r is made. In symbols, this can be expressed as:

$$C = (a, ci) = r\left(\frac{a}{r}, \frac{c}{r}\right). \tag{2.5.31}$$

The above equality is true if $r \neq 0$, so among the many possibilities for r, let us chose one in specific. Let

$$r = \|C\| \tag{2.5.32}$$

so that for every complex number $C = a + ci$, the following is always true:

$$C = (a, ci) = \|C\|\left(\frac{a}{\|C\|}, \frac{c}{\|C\|}\right) \tag{2.5.33}$$

where a, c, and $\|C\| \in \mathbb{R}$.

We can see that from the trigonometric point of view, the ratios

$$\frac{a}{\|C\|} \quad \text{and} \quad \frac{c}{\|C\|} \tag{2.5.34}$$

represent the cosine and sine of the angle Φ comprised by the sides a, c, and $\|C\|$ with vertex at the origin. Therefore, if

$$\frac{a}{\|C\|} = cos(\Phi) \quad \text{and} \quad \frac{c}{\|C\|} = sin(\Phi) \tag{2.5.35}$$

then, we can substitute the above trigonometric relations in above to obtain:

$$C = (a, ci) = \|C\|\Big(cos(\Phi), sin(\Phi)i\Big) \tag{2.5.36}$$

or, more explicitly

$$C = \|C\|\Big(cos(Arg(C), sin(Arg(C))i\Big). \tag{2.5.37}$$

The importance of this last result is that it represents another way of writing complex numbers given in rectangular coordinates into another system on which the predefined notions of norm and argument are taken into account.

This trigonometric representation of complex numbers is also called the **polar representation** of complex numbers.

Sometimes, when dealing with multiplication of complex numbers, it is useful having them trigonometrically represented. By using the previous theorems it is now possible now to prove that the multiplication of two complex numbers expressed in trigonometric form is equivalent to multiply their norms and add their arguments.

In relation to this, the following theorem is stated without proof in the chapter devoted to theorem proofs.

THEOREM 2.6. *Let C and C' be two complex numbers expressed in trigonometric form.*

$$C = \|C\|\Big(cos(Arg(C), sin(Arg(C))i\Big) \tag{2.5.38}$$

and

$$C' = \|C'\|\Big(cos(Arg(C'), sin(Arg(C'))i\Big). \tag{2.5.39}$$

Also let

$$K = \|C\| * \|C'\| \tag{2.5.40}$$

and

$$L = Arg(C) + Arg(C') \tag{2.5.41}$$

then

$$C * C' = K\Big(cos(L) + sin(L)i\Big). \tag{2.5.42}$$

Chapter 3

Transcomplex Numbers

3.1 Introduction

This chapter is short while at the same time is the heart of the *Foundations*. Is short because it relies on the preliminary definitions and theorems developed in the preceeding two chapters.

The chapter begins with the definition of transcomplex numbers as ordered pairs of ordered pairs. The operations of addition and multiplication for transcomplex numbers are introduced as an extension of those defined for the complex numbers.

Also reintroduced in this chapter are the concepts of absolute value, norm and argument, but seen from a broader perspective; so broad that the norm will result in pair of numbers. Also, the argument of a transcomplex number will result in an ordered pair of angles.

The new concepts of transnorm and transargument will be of a great help in understanding the nature of the transcomplex numbers.

Exponentiation of transcomplexs is another of the topics developed. The final objective is to prove that transcomplex numbers have the same trigonometric representation as the complex numbers, but viewed from a broader perspective.

But the core of this chapter is that if the complex numbers are "complex" by virtue of its properties and not by virtue of its entries, the transcomplex numbers ARE also complex numbers, but viewed from a broader point of view. So broad, that the complex numbers are merely a special case of the transcomplexs.

At the end, it will be proved that the transcomplex numbers also contains all the attributes of a field.

3.2 Transcomplex numbers

3.2.1 Transcomplex numbers

Now we consider an ordered pair whose elements are a real numbers ordered pair and an imaginary numbers ordered pair.

When we talk of an ordered pair constituted of the ordered pairs (a, b) and $(c, d)i$, we are conceptualizing the ordered pair (a, b) and (c, d)i as single elements of another bigger set. Thus, going to the basic definition of ordered pair, the ordered pair of the ordered pairs (a, b) and $(c, d)i$ is simply the two elements set T given by:

$$\Big\{\{(a,b)\},\{(a,b),(c,d)i\}\Big\}. \tag{3.2.1}$$

Definition 41. Let T denote an ordered pair constituted by a real numbers ordered pair (a, b) and an imaginary numbers ordered pair $(c, d)i$. We call that kind of ordered pair a **transcomplex number**. Generically, we can call the transcomplex number as T-**numbers**

$$T = (a, b) + (c, d)i. \tag{3.2.2}$$

When it is understood that the pair (a, b) is a real numbers ordered pair, and $(c, d)i$ an imaginary numbers ordered pair, the notation can be used alternatively as the following one:

$$T = (a, b, c, d). \tag{3.2.3}$$

This is the first time that the concept of transcomplex number is mentioned by its explicit attributes. Despite the rather elaborate development of this new kind of number, the surprise is that it is nothing esoteric, but, on the contrary, it is the double application of the concept of ordered pair

THEOREM 3.1. *Two transcomplex numbers $T = (a, b, c, d)$ and $T' = (a', b', c', d')$ are equal if and only if $a = a'$, $b = b'$, $c = c'$, and $d = d'$.*

Proof. (See the chapter on Theorem Proofs) □

Definition 42. Let $\mathbb{T}$ denote the **class of all transcomplex numbers**.

Now follow a simple but fundamental theorem that relates all the four numbers classes so far discussed; i.e., the reals, the image-reals, the imaginaries, and the image-imaginaries.

THEOREM 3.2. *Let T be any transcomplex number $T = (a, b, c, d)$, then*

$$T = a + b\,\tilde{} + ci + (di)\,\tilde{}. \tag{3.2.4}$$

Proof. (See the chapter on Theorem Proofs) □

Definition 43. Let $T = (a, b, c, d)$ be any transcomplex number. The **real part** of T, denoted by $Re(T)$, is the real numbers ordered pair (a, b), and the **imaginary part** of T, denoted by $Im(T)$, is the imaginary ordered pair $(c, d)i$.

In the preceding definition:

$$\begin{aligned} Re(T) &= Re(a, b, c, d) \\ &= Re\big((a, b), (c, d)i\big) \\ &= (a, b) \end{aligned} \tag{3.2.5}$$

and

$$\begin{aligned} Im(T) &= Im(a, b, c, d) \\ &= Im\big((a, b), (c, d)i\big). \\ &= (c, d)i \end{aligned} \tag{3.2.6}$$

3.3 Operations with transcomplex numbers

3.3.1 Addition and multiplication of transcomplex numbers

The addition and multiplication for transcomplex numbers will be the same as those defined for complex numbers. This will later permit us to prove that the complex numbers class is a subclass of the transcomplex numbers class and that the complex numbers field is a field like the transcomplex number field.

Definition 44. The operation of **addition**, +, and **multiplication**, *, for two transcomplex numbers $T = (a, b, c, d)$ and $T' = (a', b', c', d')$ are defined as follows:

$$T + T' = (a + a', b + b', c + c', d + d') \tag{3.3.1}$$

and

$$T * T' = (aa' - cc', bb' - dd', ac' + ca', bd' + db'). \tag{3.3.2}$$

Compare equation 3.3.1 with equation 2.5.8 to see the similarity in the definitions of addition of complex plus complex and the addition of transcomplex plus transcomplex.

Similarly, compare Eq. 3.3.2 with equations 2.5.9 and 2.5.10 to see the similarity between complex numbers and the transcomplexs and note how easy is to obtain the multiplication rule for complexs by just equating to zero the image-real and the image-imaginary entries in the above definition.

A short reflection about the nature of complex numbers is appropriate here. What is a complex number?, or, the same question stated in other terms: what makes a pair of numbers be called "complex"? Are the entries, or are the properties? Surely must be its properties, because the number $a+bi$ is not more "complex" than the real number $a+(-b)$. So, it is the addition and multiplication properties that makes a + bi more "complex" that the number $a+(-b)$.

3.3.2 The scalar product for transcomplex numbers

Definition 45. The **scalar product** of a real number r and a transcomplex number $T=(a,b,c,d)$, written as rT, is defined as follows:

$$rT = r(a,b,c,d) = (ra,rb,rc,rd). \tag{3.3.3}$$

3.3.3 Absolute value and norm of a transcomplex number

Definition 46. The **absolute value** of a transcomplex number $T=(a,b,c,d)$ is denoted by the symbol $|T|$ and defined by:

$$|T| = \big|(a,b)\big| + \big|(c,d)i\big|. \tag{3.3.4}$$

Note that the absolute value of a transcomplex number is also equivalent to say that:

$$|T| = \big|Re(T)\big| + \big|Im(T)\big|. \tag{3.3.5}$$

Furthermore, respect to the absolute value of a transcomplex number, it can be immediately seen (Recall the previous definition for the absolute value of a real ordered pair and the absolute value of an imaginary ordered pair) that:

$$\begin{aligned}|T| &= \big|(a,b)\big| + \big|(c,d)i\big| \\ &= \big(|a|,|b|\big) + \big(|c|,|d|\big)i \\ &= \big(|a|,|b|,|c|,|d|\big). \end{aligned} \tag{3.3.6}$$

and no further simplification can be done. Thus, the absolute value of a transcomplex number is again a transcomplex number, but made of positive entries.

When the transcomplex number T becomes a complex number then

$$\begin{aligned} |T| &= \big(|a|, |0|\big) + \big(|c|, |0|\big) \\ &= |a| + |c|i \end{aligned} \tag{3.3.7}$$

the same definition we previously stated for absolute value of a complex numbers.

Definition 47. The **norm** of a transcomplex number is denoted by $\|T\|$ and defined by

$$\|T\| = |\sqrt{(Re(T))^2 + (Im(T))^2}|. \tag{3.3.8}$$

For simplicity of notation we can let:

$$Re^2(T) = \big(Re(T)\big)^2 \tag{3.3.9}$$

$$Im^2(T) = \big(Im(T)\big)^2 \tag{3.3.10}$$

so that

$$\|T\| = \big|\sqrt{Re^2(T) + Im^2(T)}\big|. \tag{3.3.11}$$

With the norm of a transcomplex number, things are different than with the absolute value.

Since $Re(T) = (a, b)$ and $Im(T) = (c, d)i$, then using a previous definition for exponentiation of ordered pairs (in particular for the square of ordered pairs) we have:

$$\begin{aligned} \|T\| &= \big|\sqrt{Re^2(T) + Im^2(T)}\big| \\ &= \left|\sqrt{(a, b)^2 - \big((c, d)i\big)^2}\right| \\ &= \left|\sqrt{(a, b)^2 - \big((c, d)^2 * -1\big)}\right| \\ &= \big|\sqrt{(a, b)^2 + (c, d)^2}\big| \\ &= \big|\sqrt{(a^2, b^2) + (c^2, d^2)}\big| \\ &= \big|\sqrt{(a^2 + c^2, b^2 + d^2)}\big|. \end{aligned} \tag{3.3.12}$$

In particular for radicalization of ordered pairs, this can be further simplified to:

$$\|T\| = \big(|\sqrt{a^2 + c^2}|, |\sqrt{b^2 + d^2}|\big). \tag{3.3.13}$$

Therefore, the norm of a transcomplex number is a real numbers ordered pair.

THEOREM 3.3. *Let $T = (a, b, c, d)$ and $T' = (a', b', c', d')$. Then*

$$\|T\| * \|T'\| = \|T * T'\|. \tag{3.3.14}$$

Proof. (See the chapter on Theorem Proofs) □

3.3.4 The argument of a transcomplex number

Definition 48. The **argument** of a transcomplex number $T = (a, b, c, d)$ is denoted by $Arg(T)$ and given by:

$$Arg(T) = \tan^{-1} \frac{(c, d)}{(a, b)}. \tag{3.3.15}$$

The argument of a transcomplex number is not defined in the particular case when $a = b = 0$.

Obviously, by

$$\tan^{-1} \frac{(c, d)}{(a, b)} \tag{3.3.16}$$

we mean

$$\tan^{-1} \left(\frac{(c, d)}{(a, b)} \right) \tag{3.3.17}$$

which in turn is

$$\tan^{-1} \left(\frac{c}{a}, \frac{d}{b} \right). \tag{3.3.18}$$

Therefore,

$$\tan^{-1} \frac{(c, d)}{(a, b)} = \tan^{-1} \left(\frac{c}{a}, \frac{d}{b} \right). \tag{3.3.19}$$

THEOREM 3.4. *Let $T = (a, b, c, d)$ and $T' = (a', b', c', d')$. Then*

$$Arg(T * T') = Arg(T) + Arg(T'). \tag{3.3.20}$$

Proof. (See the chapter on Theorem Proofs) □

3.3.5 The trigonometric form of transcomplex numbers

Now we use of the concepts of norm and argument of transcomplex number already introduced and state the following theorem:

THEOREM 3.5. *Let $T = (a, b, c, d)$ be a transcomplex number. Then*

$$T = \|T\| * \Big(cos\big(Arg(T)\big), sin\big(Arg(T)\big) \Big). \tag{3.3.21}$$

Proof. (See the chapter on Theorem Proofs) □

Compare this result with the trigonometric representation of complex numbers in the preceding chapter.

3.4 The transnorm and transargument

3.4.1 Transnorm and transargument

In view of the previous results, an additional definition is needed to accomplish on the transcomplex numbers class $\mathbb{T}$ the same role that the norm does in the complex numbers $\mathbb{C}$, and the same role that the absolute value does in the class $\mathbb{R}$.

Definition 49. The **transnorm** of a transcomplex number T is denoted by ${}^{+}\|T\|$ and formulated by:

$$ {}^{+}\|T\| = \Big\| \|T\| \Big\|. \tag{3.4.1} $$

Definition 50. The **transargument** of a transcomplex number T is denoted by ${}^{+}Arg(T)$ and given by

$$ {}^{+}Arg(T) = \big\| Arg(T) \big\|. \tag{3.4.2} $$

Stated in words, a transnorm is a norm of a norm, and a transargument is the norm of an argument.

Recall by a previous theorem that the norm of a transcomplex number can be computed as

$$ \begin{aligned} \|T\| &= \left(\sqrt{a^2+c^2}, \sqrt{b^2+d^2}\right) \\ &= (r_1, r_2). \end{aligned} \tag{3.4.3} $$

Where r_1 and r_2 are real numbers. Hence,

$$ \begin{aligned} {}^{+}\|T\| &= \Big\| \|T\| \Big\| \\ &= \|(r_1, r_2)\|. \end{aligned} \tag{3.4.4} $$

But, the norm of an ordered pair is a positive real number. Then,

$$ \begin{aligned} {}^{+}\|T\| &= \|(r_1, r_2)\| \\ &= \left\| \sqrt{r_1^2 + r_2^2} \right\| \end{aligned} \tag{3.4.5} $$

$$ \begin{aligned} &= \left\| \sqrt{\left(|\sqrt{a^2+c^2}|\right)^2 + \left(|\sqrt{b^2+d^2}|\right)^2} \right\| \\ &= |\sqrt{a^2+b^2+c^2+d^2}|. \end{aligned} \tag{3.4.6} $$

On the other hand, the concept of transargument was defined by finding the norm of an argument. The symmetry of two new concepts —transnorm and transargument— can be depicted as follows:

$$\text{transnorm} = \text{norm of a norm}$$
$$\text{transargument} = \text{norm of an argument.}$$

Thus,

$$^{+}Arg(T) = \|(T)\| = \|(\Theta_1, \Theta_2)\|. \tag{3.4.7}$$

But since (Θ_1, Θ_2) is an ordered pair, its norm is computed as the definition of norm states. Therefore,

$$\begin{aligned} ^{+}Arg(T) &= \|Arg(T)\| \\ &= \left\|(\Theta_1, \Theta_2)\right\| \\ &= \left\|\sqrt{(\Theta_1^2, \Theta_2^2)}\right\| \\ &= \left|\sqrt{\left(\tan^{-1}\frac{c}{a} + \tan^{-1}\frac{d}{b}\right)}\right|. \end{aligned} \tag{3.4.8}$$

THEOREM 3.6. *Let $T = (a, b, c, d)$ and $T = (a', b', c', d')$. Then,*

$$^{+}\|T\| \; * \; ^{+}\|T'\| \; \neq \; ^{+}\|T * T'\|. \tag{3.4.9}$$

Proof. (See the chapter on Theorem Proofs) □

THEOREM 3.7. *Let $T = (a, b, c, d)$ and $T = (a', b', c', d')$. Then,*

$$^{+}Arg(T) \; * \; ^{+}Arg(T') \neq \; ^{+}Arg(T) +^{+} Arg(T'). \tag{3.4.10}$$

Proof. (See the chapter on Theorem Proofs) □

The last two theorems are important for future developments.

3.5 Exponentiation of transcomplex numbers

3.5.1 The exponential form of transcomplex numbers

THEOREM 3.8. *Let $T = (a, b, c, d)$ be a transcomplex number. For short, let*

$$R =^{+} \left\|\|T\|\right\| = (r_1, r_2) \tag{3.5.1}$$

and

$$W =^{+} Arg(T) = \left\|Arg(T)\right\| = (\Theta_1, \Theta_2) \tag{3.5.2}$$

then

$$T = Re^{iW} \tag{3.5.3}$$

where e is the natural logarithm base and i the imaginary unit.

Proof. (See the chapter on Theorem Proofs) □

3.6 Other operation with transcomplex numbers

In view of the preceding theorems and definitions, now it is possible to state for the transcomplex numbers the same operations that applies for the complex numbers.

3.6.1 Multiplication and division of transcomplex numbers

Definition 51. For two transcomplex number T and T', and using the exponential notation already described, the **multiplication** of T by T' is simply:

$$T * T' = R * R' e^{i(W+W')} \tag{3.6.1}$$

or

$$T * T' =^{+} \|T\| \ * \ ^{+}\|T'\| e^{i\left(Arg(T)+Arg(t')\right)}. \tag{3.6.2}$$

The **division** of T by T' is given by:

$$\frac{T}{T'} = \frac{R}{R'} e^{i(W-W')} \tag{3.6.3}$$

or

$$\frac{T}{T'} = \frac{^{+}\|T\|}{^{+}\|T'\|} e^{i\left(Arg(T)-Arg(t')\right)}. \tag{3.6.4}$$

3.6.2 Transcomplex exponentiation of transcomplex numbers

Before introducing the transcomplex exponentiation, the following results are important to be stated.

Definition 52. The **real exponentiation** of a transcomplex number T to a real number n is given by:

$$T^n = R^n e^{inW} \tag{3.6.5}$$

or

$$T^n =^+ \|T\|^n e^{inArg(T)} \tag{3.6.6}$$

and

$$T^{-n} = R^{-n} e^{-inW} \tag{3.6.7}$$

or

$$T^{-n} =^+ \|T\|^{-n} e^{-inArg(T)}. \tag{3.6.8}$$

3.6.3 Transcomplex exponents

Transcomplex numbers can be raised to transcomplex powers.

Definition 53. Let T and T' be two transcomplex numbers, that is, let:

$$T = Re^{iW} \qquad \text{and} \qquad T' = R'e^{iW'} \tag{3.6.9}$$

then the **transcomplex exponentiation** of the transcomplex number T to the **transcomplex exponent** T' is:

$$T^{T'} = \left(Re^{iW}\right)^{R'e^{iW'}}. \tag{3.6.10}$$

3.7 The transcomplex numbers field

An important goal of this book is to prove that the transcomplex number makes a field, as the real numbers and the complex numbers do.

THEOREM 3.9. *The transcomplex number class $\mathbb{T}$, together with the addition and multiplication operations defined for this class make a field.*

Proof. (See the chapter on Theorem Proofs) □

Chapter 4

The Coordinate System S^4

4.1 Introduction

The essential concepts toward a consistent theory of transcomplex numbers have already being laid. But that is not the whole goal, because a transcomplex number cannot be plotted on a coordinate system using the usual axis labeling.

A new paradigm of complex variables needs also a new paradigm to plot its functions. The new coordinate system S^4, in harmony with the transcomplex number system, is the idoneous place to plot its transcomplex functions.

In order to plot the transcomplex numbers no axis can be identical with the others. For that reason, the transcomplex space contains one real axis, one image-real axis, (not identical with the real-axis), one imaginary numbers axis (not identical with the imaginary numbers neither with the image-real axis), and a fourth image-imaginary axis (not identical with any of the preceding ones). The creation of a new type of tetraxial space with a new labeling for each axis is what this chapter is about. But more than that, S^4 is a formal structure where all axes have a sequence number, the many tridimensional spaces contained within S^4 also have a spatial sequencing, and each octant within each subspace is also uniquely numbered.

A four-dimensioned space is impossible to grasp in our minds, much less the behavior of functions in that hyperspace. But, by dividing the tetraspace into 4 cubicles—here called subspaces—and further dividing the cubicles into octants, the challenge is attained, or at least is nearer to our intellectual understanding. All this is needed to give to the "fourth dimension" the formality it has always lacked. There has been no standardization as to what the "second" dimension means, much less as to what the "fourth dimension" stands for. The reason behind to say that even the "second" dimension is not standardized is that for "second" we mean what follows the "first". However, in the geometry of the space S^4, the second dimension is the $Y\tilde{}$-axis no matter from where one begins to count.

Hence, S^4 is an ordered hyperspace, and not merely a 4-dimensioned space, much less a geometrical curiosity.

Numbering the octants and axes will result fruitful when mentally joining the tridimensional pictures.

Finally, a table is given depicting every possible type of transcomplex number; 16 in total. Under the usual complex number system, we can derive only three possibilities: real numbers, imaginary numbers, or complex numbers. From the transcomplexs, we can derive 13 additional cases. A short reflection on the fact that transcomplexs are also complexs will lead us to the conclusion that with the complex numbers alone we have been missing the mayor part of the picture.

4.2 The Space S^4

4.2.1 The tetraxial hyperspace

Definition 54. The **space** S^4, or more clearly, the **tetraspace** S^4, is the universe set of the transcomplex numbers. Each element of S^4, also called a **point** of S^4, is a 4-tuple (a, b, c, d). In symbols:

$$S^4 = \{(a, b, c, d) \mid a \in X,\ b \in Y\tilde{},\ c \in iZ,\ d \in iZ\tilde{}\}. \tag{4.2.1}$$

In other words, the tetraspace S^4 is also the class of all transcomplex numbers $\mathbb{T}$.

Definition 55. In a transcomplex number $T = (a, b, c, d)$ we call a the **X-coordinate**, b the **Y˜-coordinate**, c the **iZ-coordinate**, and d the **i˜Z -coordinate** of T respectively. The **unitary elements** for each one of the axes are:

$$\begin{aligned} 1 &= (1, 0, 0, 0) \\ 1\tilde{} &= (0, 1, 0, 0) \\ i &= (0, 0, 1, 0) \\ i\tilde{} &= (0, 0, 0, 1). \end{aligned} \tag{4.2.2}$$

Similarly, and by virtue of the scalar product, any other number a lying on any of the four axes is also a 4-tuple, as shown below:

$$\begin{aligned} a &= a(1,0,0,0) &&= (a,0,0,0) \\ b\tilde{} &= b(0,1,0,0) &&= (0,b,0,0) \\ ci &= c(0,0,1,0) &&= (0,0,c,0) \\ di\tilde{} &= d(0,0,0,1) &&= (0,0,0,d). \end{aligned} \tag{4.2.3}$$

Therefore, a point $T = (a, b, c, d)$ in the hyperspace S^4 contains its corresponding coordinates on each of the four axes at the same time:

$$T = a1 + b1\tilde{} + ic + i\tilde{}d. \tag{4.2.4}$$

If the last entries of a transcomplex number are zero, they can be dropped for simplification of reading, as:

$$\begin{aligned} a &= (a,0,0,0) \\ (a,b) &= (a,b,0,0) \\ (a,b,c) &= (a,b,c,0). \end{aligned} \tag{4.2.5}$$

Definition 56. The transcomplex numbers set

$$B = \{1, 1\tilde{}, i, i\tilde{}\}. \tag{4.2.6}$$

is called the **basis** of the tetraspace S^4, and each one of the elements of this set will be called an **axial unit**.

Below is a complete table of all possible outcomes of multiplication of axial units. Although the multiplication of transcomplex numbers is commutative, the commuted products are also shown.

Factor 1	Factor 2	Product
1 = (1, 0, 0, 0)	1 = (1, 0, 0, 0)	1 = (1, 0, 0, 0)
1 = (1, 0, 0, 0)	1˜= (0, 1, 0, 0)	0 = (0, 0, 0, 0)
1 = (1, 0, 0, 0)	i = (0, 0, 1, 0)	i = (0, 0, 1, 0)
1 = (1, 0, 0, 0)	i˜= (0, 0, 0, 1)	0 = (0, 0, 0, 0)
1˜= (0, 1, 0, 0)	1 = (1, 0, 0, 0)	0 = (0, 0, 0, 0)
1˜= (0, 1, 0, 0)	1˜= (0, 1, 0, 0)	1˜= (0, 1, 0, 0)
1˜= (0, 1, 0, 0)	i = (0, 0, 1, 0	0 = (0, 0, 0, 0)
1˜= (0, 1, 0, 0)	i˜= (0, 0, 0, 1)	i˜= (0, 0, 0, 1)
i = (0, 0, 1, 0)	1 = (1, 0, 0, 0)	i = (0, 0, 1, 0)
i = (0, 0, 1, 0)	1˜= (0, 1, 0, 0)	0 = (0, 0, 0, 0)
i = (0, 0, 1, 0)	i = (0, 0, 1, 0)	-1 = (-1, 0, 0, 0)
i = (0, 0, 1, 0)	i˜= (0, 0, 0, 1)	0 = (0, 0, 0, 0)
i˜= (0, 0, 0, 1)	1 = (1, 0, 0, 0)	0 = (0, 0, 0, 0)
i˜= (0, 0, 0, 1)	1˜= (0, 1, 0, 0)	i˜= (0, 0, 0, 1)
i˜= (0, 0, 0, 1)	i = (0, 0, 1, 0)	0 = (0, 0, 0, 0)
i˜= (0, 0, 0, 1)	i˜= (0, 0, 0, 1)	-1˜= (0, -1, 0, 0)

Table 4.1: All possible outcomes of multiplication of axial units

A simplified version of this table is the following:

Factor 1	Factor 2	Product
1	1	1
1	$1^{\sim}$	0
1	i	i
1	$i^{\sim}$	0
$1^{\sim}$	1	0
$1^{\sim}$	$1^{\sim}$	$1^{\sim}$
$1^{\sim}$	i	0
$1^{\sim}$	$i^{\sim}$	$i^{\sim}$
i	1	i
i	$1^{\sim}$	0
i	i	-1
i	$i^{\sim}$	0
$i^{\sim}$	1	0
$i^{\sim}$	$1^{\sim}$	$i^{\sim}$
$i^{\sim}$	i	0
$i^{\sim}$	$i^{\sim}$	$-1^{\sim}$

Table 4.2: Simplified form for all possible outcomes of multiplication of axial units

The products equal to zero are:

$$\begin{aligned} 1 * 1^{\sim} &= 0 \\ 1 * i^{\sim} &= 0 \\ 1^{\sim} * i &= 0 \\ i * i^{\sim} &= 0. \end{aligned} \tag{4.2.7}$$

and the non-zero products whose real part is zero are:

$$\begin{aligned} 1 * i &= (0, 0, i, 0) \\ 1^{\sim} * i^{\sim} &= (0, 0, 0, 1). \end{aligned} \tag{4.2.8}$$

From this we can note that the above 6 products are precisely the possible products of one axis times another.

We can use those products of the axial units to define perpendicularity of axes, as follows

Definition 57. Two transcomplex numbers U and V are said to be **mutually perpendicular**, and denoted by $U \perp V$, if the real part of their product is zero. Two sets $\mathbb{U}$ and $\mathbb{V}$ are said to be **mutually orthogonal**, denoted by $\mathbb{U} \perp \mathbb{V}$, if every element of $\mathbb{U}$ is perpendicular to every element of $\mathbb{V}$.

It will be now shown that in view of this definition, all of the four axes of the tetraspace S^4 are mutually orthogonal. This is stated in the following theorem:

THEOREM 4.1. *Respect to the orthogonality of the axes of the space S^4 the following two propositions are true:*

- *The four axes of the tetraspace S^4 are mutually orthogonal*
- *No axis is orthogonal to itself.*

Proof. (See the chapter on Theorem Proofs) □

THEOREM 4.2. *No element of one axis can be plotted into another axis.*

Proof. (See the chapter on Theorem Proofs) □

Simple as it seems, what the theorem states is that the axes are not interchangeable, implying that the system S^4 is an ordered tetraspace.

Definition 58. If in a transcomplex number its last coordinate is zero, we say that it is a **real transcomplex number**, otherwise we call it a **complete transcomplex**.

According to this definition, the reals, the image-reals, the real ordered pairs, the imaginaries and the complex numbers are all real transcomplex numbers, because in all of them the last coordinate is zero.

4.3 The Subspaces of S^4

4.3.1 The six basic planes of the S^4 space

The way the axes were chosen resulted in a right-handed coordinate axial system. When in a transcomplex number $T = (a, b, c, d)$ it is chosen $c = d = 0$ in the imaginary iZ and $i\tilde{}$Z-axes we obtain the common $XY\tilde{}$plane.

Similarly, we obtain other planes when other pairs of coordinates are chosen to be zero.

In total, there are 6 basic planes on the hyperspace S^4 which are defined below.

Definition 59. The six basic planes generated by pairs of axes of the space S^4 are:

1. The **normal plane**: generated by the X and $Y\tilde{}$axes and usually called the XY-plane. Any point on this plane is of the form

$$p = (a, b, 0, 0). \tag{4.3.1}$$

If the normal plane is displaced a distance δ along the iZ-axis, then its equation is

$$p = (a, b, \delta, 0). \tag{4.3.2}$$

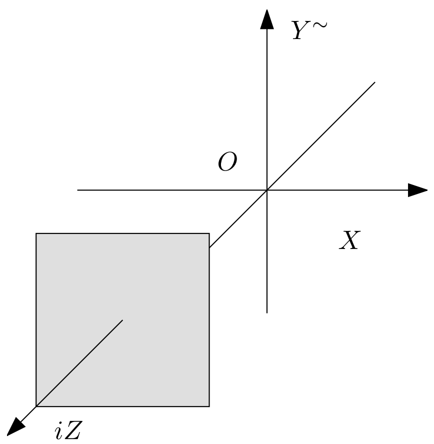

Figure 4.1: The normal plane shown displaced along the iZ-axis

2. The **osculating plane**: generated by the X and the iZ axes. The osculating plane is the plane of the complex numbers $a + ic$ when it is not displaced. Any point on this plane is of the form

$$p = (a, 0, c, 0). \tag{4.3.3}$$

If the osculating plane is displaced a distance δ along the $Y\tilde{}$-axis, then its equation is

$$p = (a, \delta, c, 0). \tag{4.3.4}$$

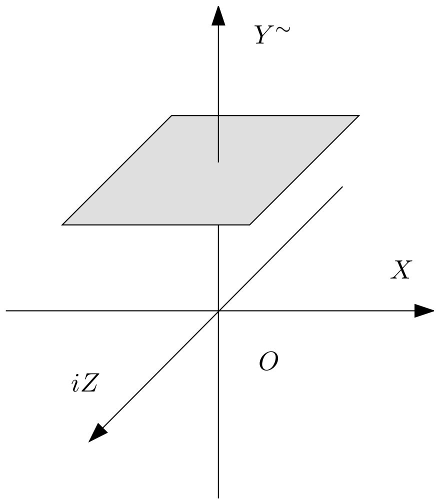

Figure 4.2: The osculating plane shown displaced along the image-reals axis

3. The **rectifying plane**: generated by the $Y\tilde{}Z$ and the iZ axes. This is the plane of the vector-type complexs. The product of two points on this plane does not belong to this plane, but to the normal plane. Points on this plane are of the form

$$p = (0, b, c, 0). \tag{4.3.5}$$

If the rectifying plane is displaced a distance δ along the X-axis, then its equation is

$$p = (\delta, b, c, 0). \tag{4.3.6}$$

4. The **imaginary osculating plane**: generated by the iZ and $i\tilde{}$Z axes. On this plane are the imaginary ordered pairs. All of them are of the form

$$p = (0, 0, c, d). \tag{4.3.7}$$

If the imaginary osculating plane is displaced a distance δ along the $Y\tilde{}$-axis, then its equation is

$$p = (0, \delta, c, d). \tag{4.3.8}$$

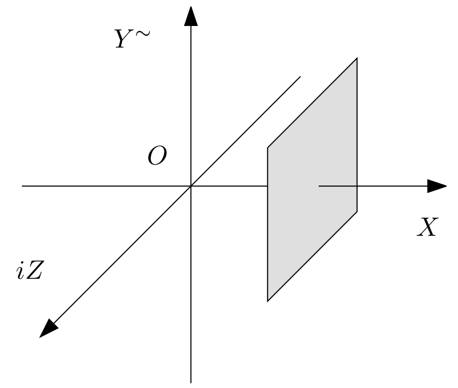

Figure 4.3: The rectifying plane shown displaced along the X-axis

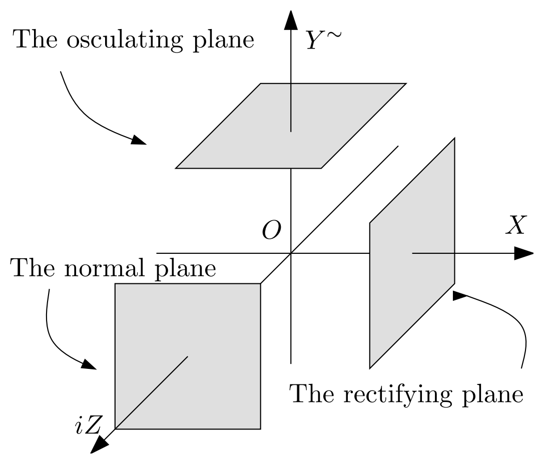

Figure 4.4: The real planes

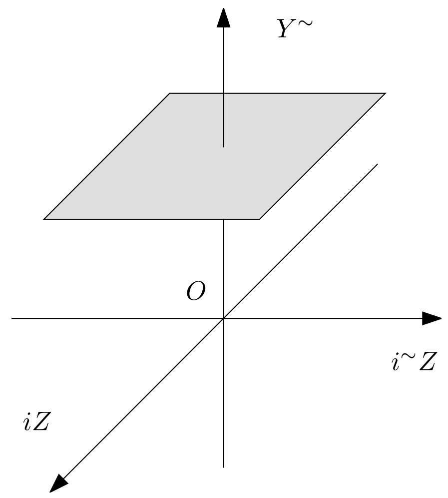

Figure 4.5: The imaginary osculating plane shown displaced along the image-imaginary axis

5. The **imaginary rectifying plane**: generated by the $Y\tilde{}$ and iZ axes. Points on this plane are of the form

$$p = (0, b, 0, d). \tag{4.3.9}$$

If the imaginary rectifying plane is displaced a distance δ along the X-axis, then its equation is

$$p = (\delta, b, 0, d). \tag{4.3.10}$$

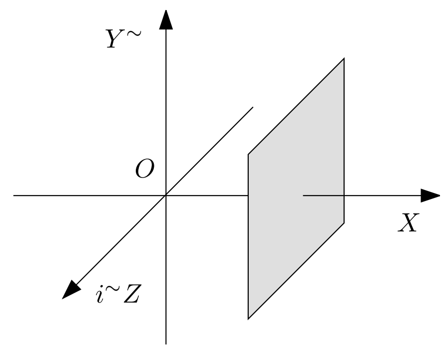

Figure 4.6: The imaginary rectifying plane shown displaced along the X-axis

6. The **imaginary normal plane**: generated by the X and the $i\tilde{}$Z axes. They are of the form

$$p = (a, 0, 0, d). \tag{4.3.11}$$

If the imaginary normal plane is displaced a distance δ along the $Y\tilde{}$-axis, then its equation is

$$p = (a, \delta, 0, d). \tag{4.3.12}$$

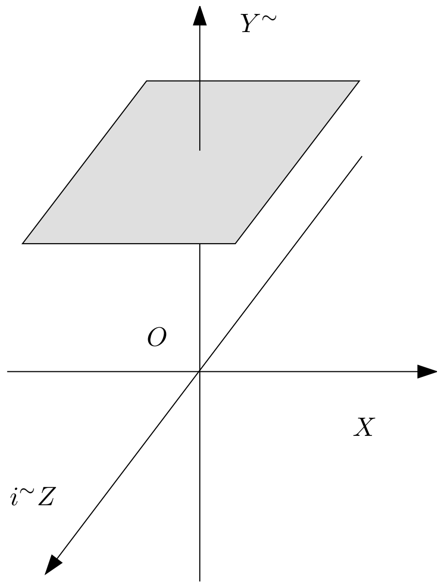

Figure 4.7: The imaginary normal plane shown displaced along the image-reals axis

Planes parallel to any of the basic planes of S^4 have simple equations to describe them. Thus, a plane p parallel to the $XY\tilde{}$-plane will have as equations iZ = constant and $i\tilde{}$Z = 0. In set theory notation, that plane could be described as

$$p = \big\{(a, b, c, d)|\ c = constant,\ d = 0\big\}. \tag{4.3.13}$$

All the planes defined occupy 2 axes simultaneously, but keep in mind that since there are four axes in S^4, then each plane has 2-degrees of freedom (the two axes not occupied). In the $XY\tilde{}iZ$ subspace, for example, that means that the normal plane can be moved along the iZ as illustrated, but also can be moved as displaced along the $i\tilde{}$Z-axis (not shown in that illustration).

4.3.2 The coordinates and subspaces numbering

It is obviously easier to visualize on paper a 3-axes system than a 4-axes one, but nothing impedes us to represent many three axes at a time, taking us mentally nearer to what a real hyperspace is.

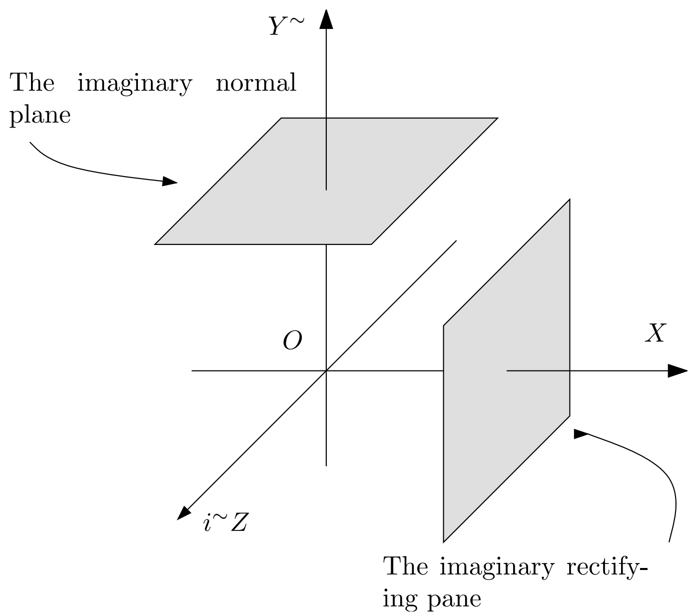

Figure 4.8: The imaginary normal and imaginary rectifying planes

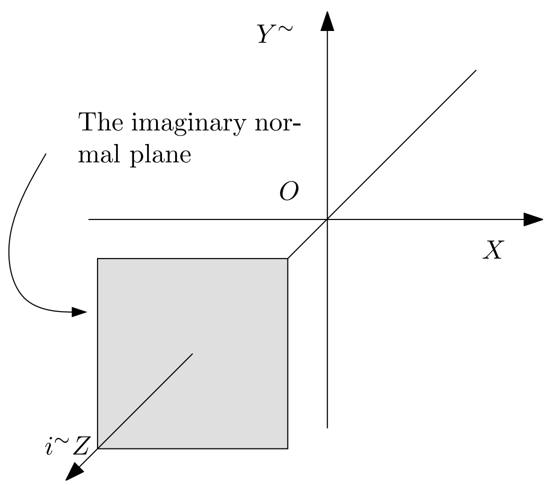

Figure 4.9: The imaginary osculating plane

There are four ways to choose three objects from a set of four; in the same way, there are 4 ways to make 3-axes subspaces from the 4-axes space S^4.

That also means that there are 4 different 3-dimensional surfaces for each transcomplex function to plot in the transcomplex range. Later we'll go over this.

Definition 60. Any three axes representation of the four that the hyperspace S^4 have will be called a **3/4 representation of** S^4, or simply a **3/4 space**, or with a single notation, an S^3 **space** or a **subspace of** S^4.

The real cubicle and the 3 hypercubicles of S^4 are shown below.

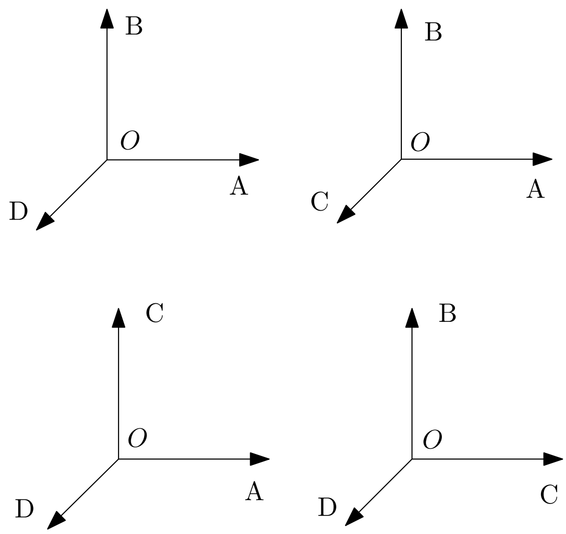

Figure 4.10: The 4 subspaces of S^4

Definition 61. The space enclosed by any S^3 subspace of S^4 will be called a **cubicle** of S^4. The space enclosed by the X, Y˜and iZ axes will be specially called the **real cubicle**, all others will be called **hypercubicles** of S^4.

Keep in mind that when we speak of the real cubicle, we are not referring to a space of three real-axes system.

In order to structure our space S^4 as much as possible, we will not be satisfied with labeling the axes by letters, but will also use numbers as follows:

Definition 62. The **coordinate number** of an axis, denoted by Co#(*axis-name*), is a natural number associated with each coordinate of S^4 as follows:

$$\begin{aligned} Co\#(X) &= 1 \\ Co\#(Y\tilde{}) &= 2 \\ Co\#(iZ) &= 3 \\ Co\#(i\tilde{}Z) &= 4. \end{aligned} \tag{4.3.14}$$

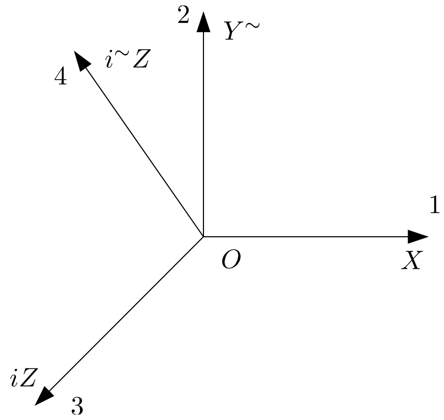

Figure 4.11: The sequential numbering of the the axes of S^4

Similarly, the cubicles will be also numbered.

Definition 63. The **cubicle number** of a triaxial subspace ($V1$, $V2$, $V3$) denoted by $Q\#(V1, V2, V3)$, is a natural number given by the rule:

$$Q\#(V1, V2, V3) = Co\#(V1) + Co\#(V2) + Co\#(V3) - 5. \tag{4.3.15}$$

The cubicle number is uniquely determined. Thus for the subspace $XY\tilde{}iZ$ we have:

$$\begin{aligned} Q\#(XY\tilde{}iZ) &= Co\#(X) + Co\#(Y\tilde{}) + Co\#(iZ) - 5 \\ &= 1 + 2 + 3 - 5 \\ &= 1. \end{aligned} \tag{4.3.16}$$

Similarly, for the subspace $XY\tilde{}$i˜Z:

$$\begin{aligned} Q\#(XY\tilde{}i\tilde{}Z) &= Co\#(X) + Co\#(Y\tilde{}) + Co\#(i) - 5 \\ &= 1 + 2 + 4 - 5 \\ &= 2. \end{aligned} \tag{4.3.17}$$

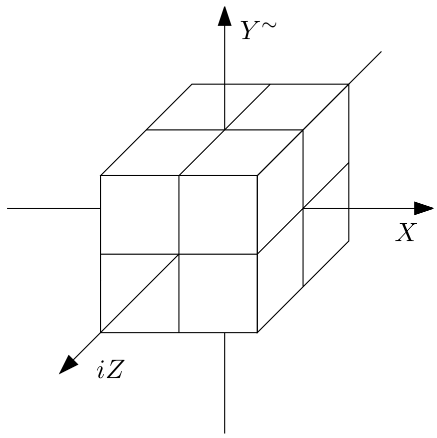

Figure 4.12: Cubicle 1 is the subspace $XY^\sim iZ$

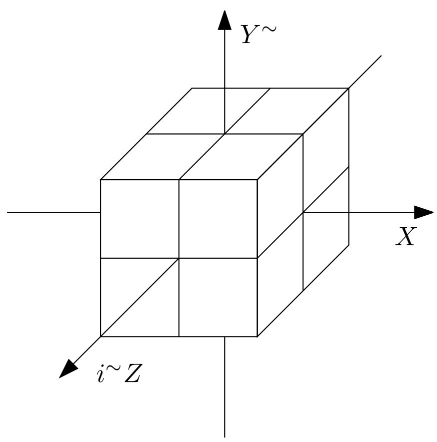

Figure 4.13: Cubicle 2 is the subspace $XY^\sim i^\sim Z$

The cubicle number of the subspace $XiZi\tilde{}Z$ is:

$$\begin{aligned} Q\#(XiZi\tilde{}Z) &= Co\#(X) + Co\#(iZ) + Co\#(i\tilde{}Z) \\ &= 1 + 3 + 4 - 5 \\ &= 3. \end{aligned} \tag{4.3.18}$$

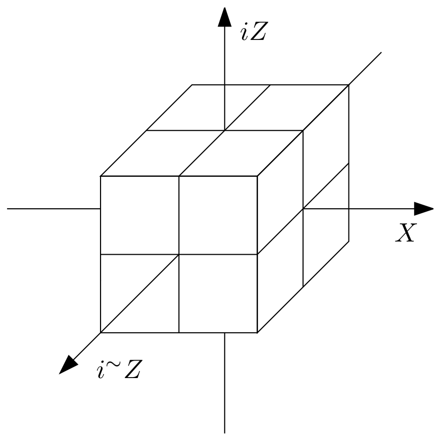

Figure 4.14: Cubicle 3 is the subspace $XiZi\tilde{}Z$

For the subspace $Y\tilde{}iZi\tilde{}Z$ the cubicle number is:

$$\begin{aligned} Q\#(Y\tilde{}iZi\tilde{}Z) &= Co\#(Y\tilde{}) + Co\#(iZ) + Co\#(i\tilde{}Z) \\ &= 2 + 3 + 4 - 5 \\ &= 4. \end{aligned} \tag{4.3.19}$$

Each cubicle of S^4 is made of 3 axes, and each axis can be positive or negative; so there are 8 octants or subcubicles per cubicle. Since there are 4 cubicles in S^4, there are a total of 32 octants in the hyperspace S^4.

The numeration or labeling of those 32 octants is arbitrary like the numbering of the quadrants on the $XY\tilde{}$-coordinates. So we'll use the following:

Definition 64. The **octant number** of a subcubicle of axes $V1$, $V2$, $V3$ is given by:

$$Oct\#(V_1^{\pm}, V_2^{\pm}, V_3^{\pm}) = 2^2\mu_1 + 2\mu_2 + \mu_3 + 2^3(Q\#(V_1, V_2, V_3) - 1) + 1 \tag{4.3.20}$$

where

$$\mu_i = \begin{cases} 0, & \text{if} \quad V_i \geq 0 \\ 1, & \text{if} \quad V_i < 0. \end{cases} \tag{4.3.21}$$

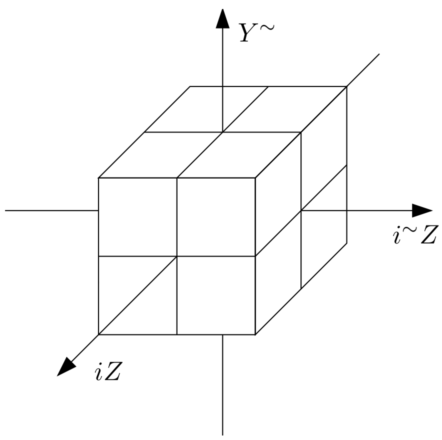

Figure 4.15: Cubicle 4 is the subspace $Y^{\sim}iZi^{\sim}Z$

In this way,

$$\begin{aligned} Oct\#(X^+, Y^{\sim +}, iZ^+) &= 2^2(0) + 2(0) + (0) + 2^3(Q\#(X, Y, iZ) - 1) + 1 \\ &= 0 + 0 + 0 + 8(1 - 1) + 1 \\ &= 1. \end{aligned} \tag{4.3.22}$$

This octant is show below.

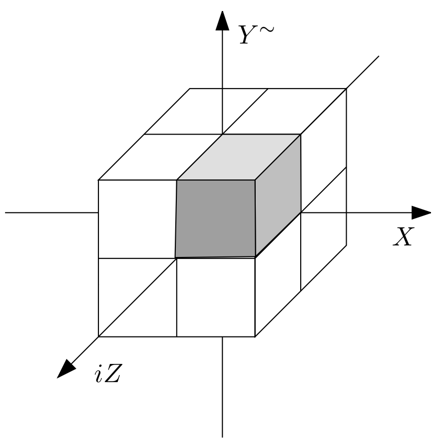

Figure 4.16: The octant 1 of the cubicle 1

As another example, the octant number comprised by the X^-, $Y^{\sim -}$ and iZ^- axes is

$$\begin{aligned} Oct\#(X^-, Y^{\sim -}, iZ^-) &= 2^2(1) + 2(1) + 2(1) + 2^3(Q\#(X, Y^\sim, iZ) - 1) + 1 \\ &= 4 + 2 + 1 + 8(1 - 1) + 1 \\ &= 7 + 0 + 1 \\ &= 8. \end{aligned} \tag{4.3.23}$$

4.3.3 The 16 types of transcomplex numbers

Each transcomplex number T have exactly four coordinates, one is real, one is image-real, one is imaginary, and the last one is image-imaginary. But not every transcomplex number is always dressed with its four entries. The transcomplex number $(0, 0, 0, 0)$ is the origin, $(0, 0, c, 0)$ is an imaginary number c, and so forth. At the end, there are 16 possibilities or types of transcomplexs. Since we are going to use frequently the concept of "type of transcomplex", it is better to define it first.

Definition 65. Let T be any transcomplex number (a, b, c, d). The **type** of the transcomplex number T, denoted by $Ty\#(T)$, is the base 10 value of a base 2 four digits number $d_1d_2d_3d_4$ associated with T by the following rule:

$$Ty\#(T) = d_1 2^3 + d_2 2^2 + d_3 2^1 + d_4 \tag{4.3.24}$$

where

$$d_i = \begin{cases} 0, & \text{if} \quad V_i = 0 \\ 1, & \text{if} \quad V_i \neq 0. \end{cases} \tag{4.3.25}$$

for $i = 1, 2, 3, 4$; and where V_i is the i-th coordinate of T.

For example, the transcomplex number $(0, b, c, 0)$ is type 6 because

$$d_1 = 0 \qquad \text{since the first coordinate of } T \text{ is } 0 \tag{4.3.26}$$

$$d_2 = 1 \qquad \text{since the second coordinate of } T \text{ is not } 0 \tag{4.3.27}$$

$$d_3 = 1 \qquad \text{since the third coordinate of T is not 0} \tag{4.3.28}$$

$$d_4 = 0 \qquad \text{since the fourth coordinate of T is 0} \tag{4.3.29}$$

hence,

$$Ty\#(T) = d_1 * 2^3 + d_2 * 2^2 + d_3 * 2^1 + d_4 * 2^0 \tag{4.3.30}$$

$$= 0 * 2^3 + 1 * 2^2 + 1 * 2^1 + 0 * 2^0 \tag{4.3.31}$$

$$= 0 + 4 + 2 + 0 \tag{4.3.32}$$

$$= 6. \tag{4.3.33}$$

Next is a table of all the transcomplex number types, and how we ordinarily visualize them. Of the 16 types, 8 of them are real transcomplexs (those with the fourth coordinate equal to 0).

Not every type of transcomplex number is closed under multiplication, although the transcomplexs as a whole are closed. For example, the transcomplexs type 2 -the imaginary numbers- are not closed under multiplication. That means that the multiplication of two transcomplexs type 2 is not again a transcomplex type 2 (that is, the multiplication of two imaginary numbers is not again an imaginary number).

The same thing happens with the transcomplexs type 6. In this case the product is another number that is on a plane perpendicular to the plane holding the multiplicands. Since in applied physics, this is the behavior of the vector product under multiplication, we will use the same name for this type of transcomplex number.

Definition 66. The transcomplex numbers of the type 3, type 6, 9, and 12, i.e., those of the form T = (0, 0, c, d), T = (0, b, c, 0), T = (a, 0, 0, d), and T = (a, b, 0, 0) respectively, will be called **vectors**.

Transcomplex number	Simplified	What is What they make	Type
(0, 0, 0, 0)	0	The origin	0
(0, 0, 0, d)	$(id)^{\sim}$	The image-imaginary numbers	1
(0, 0, c, 0)	ic	The imaginary numbers	2
(0, 0, c, d)	$ic + (id)^{\sim}$	Vectors	
		The imaginary osculating plane	
		The imaginary ordered pairs	3
(0, b, 0, 0)	$b^{\sim}$	The image real axis	4
(0, b, 0, d)	$b^{\sim} + id^{\sim}$	The image-complex numbers	5
(0, b, c, 0)	$b^{\sim} + ic$	Vectors	
		The rectifying plane	
		The semicomplex plane	6
(0, b, c, d)			7
(a, 0, 0, 0)	a	The real axis	8
(a, 0, 0, d)		Vectors	
		The imaginary normal plane	9
(a, 0, c, 0)	a + ic	The complex numbers plane	10
(a, 0, c, d)			11
(a, b, 0, 0)	(a, b)	Vectors	
		The real ordered pairs	12
(a, b, 0, d)			13
(a, b, c, 0)	(a, b, c)	The real transcomplex space	14
(a, b, c, d)	S^4	The transcomplex space	
		The tetradimensional space	15

Table 4.3: The 16 Types of Transcomplex Numbers

Chapter 5

Transcomplex Functions

5.1 Introduction

The cornerstones for the Theory of Transcomplex Numbers with its consistency and completeness is near end. This chapter, apart from being the last step in the stair of new definitions and theorems, is also about the mathematical justification for plotting complex functions into a transcomplex space.

The conventional methods of mapping complex functions will gradually disappear, giving rise to the new concept of image-mappings, semicomplex and transcomplex mappings. The plotting of functions of 2-dimensional complex variables into a 3-dimensional space will come out to be surprisingly easy.

5.2 Complex functions

5.2.1 Mappings

Definition 67. A **map**, also called a **function** or **transformation**, and symbolized by small letters such as f, g, etc., is a unique rule from a set A to a set B that assign each element a of A to exactly one element b of B. This is written as

$$A \xrightarrow{f} B \tag{5.2.1}$$

or

$$b \in B = f(a \in A) \tag{5.2.2}$$

and in simpler notation,

$$b = f(A). \tag{5.2.3}$$

The element b of B is called the **mate** under f of the element a of A. The subset of B of all mates of a map f is called the **range** of f. The subset of A of all the elements of A used by the rule f is called the **domain** of f.

Probably, the most used terminology for the concept of mate is the word *image* instead of the word *mate*, for example: *b is the image of a under the map f*. But since the word *image* has been consistently used in another context throughout this book, we'll use the word **mate** in order to avoid confusion. Mate is also used in mathematics by other authors to describe the same concept of image of an element under a function.

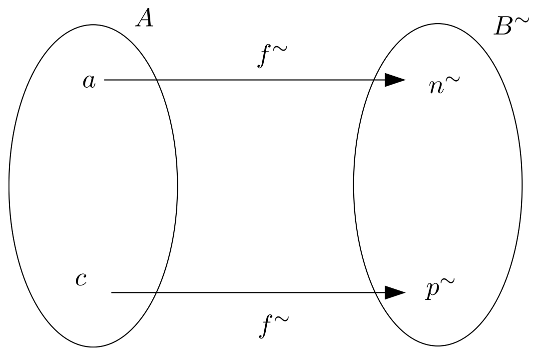

Figure 5.1: A map can relate different elements of a set A to the same element of a set B

According to the preceding definition, an element of a domain cannot have two different mates, however, several elements (or even all) of a domain may map to the same element of a range.

In the illustration above, $m = f(a)$, $m = f(b)$, and $p = f(c)$. That makes f a function in the sense of the stated definition, because no element of the set A is assigned to two different elements of the set B.

In the rule f were defined in such a way that $m = f(a)$ and $n = f(a)$ then f could not be considered a map because we would have two different mates for the same element a of the domain B.

Definition 68. If f is a map from a set A to a set B, and if the range of f consists of the entire set B, then we say that f is a mapping from A **onto** B.

Note that every "onto" mapping is an "into" mapping, but the converse is not necessarily true. The relevance of "onto" mappings is that every element of the range of a function is the mate of at least one element of its domain.

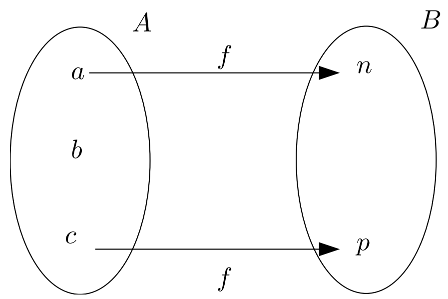

Figure 5.2: A map is onto if every element of the B is the mate of at least one element of the set A

Strictly speaking, a function is a collection (and therefore, a set) of ordered pairs $(a, f(a))$ such that if the ordered pair $(a, f(a))$ and $(b, f(b))$ belong to f then $a = b$ and $f(a) = f(b)$. Since functions are sets, two functions are equal if they contain the same elements (that is, the same collection of ordered pairs). If b is the mate of a under f, then, using the ordered pairs notation this can be written as

$$(a, b) \in f \tag{5.2.4}$$

or

$$(a, f(a)) \in f. \tag{5.2.5}$$

Definition 69. Let f be a function from a set A to a set B, then the **inverse map** f^{-1} is the map from B to A such that if

$$b \in B = f(a \in A) \tag{5.2.6}$$

then

$$a \in A = f^{-1}(b \in B). \tag{5.2.7}$$

Note from the previous figure that not every map may have an inverse because if $m = f(a)$ and $m = f(b)$ then the inverse of f maps the elements a and b to m, and that breaks with the definition of map.

Definition 70. A function is said to be **one-to-one**, denoted by $(1-1)$, if for every two distinct elements of its domain, the function assigns two distinct elements as mates. In symbols, f is $(1-1)$ if and only if

$$f(a) = b \quad \text{and} \quad f(a') = b'$$

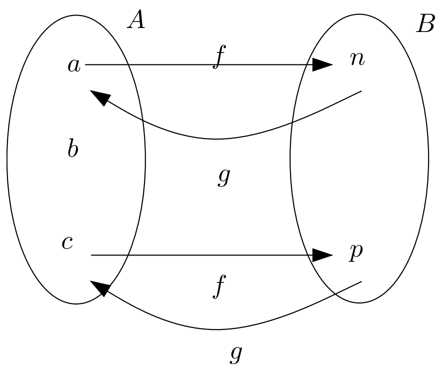

Figure 5.3: The map g is the inverse of the map f, but g is not onto, while f is

then

$$\text{then} \quad a \neq a' \quad \text{and} \quad b \neq b'. \tag{5.2.8}$$

An $(1-1)$ mapping is also called an injective mapping or an injection. If a map is injective and also onto, then that map is said to be bijective.

It can be proved that the inverse of a $(1-1)$ function is also $(1-1)$.

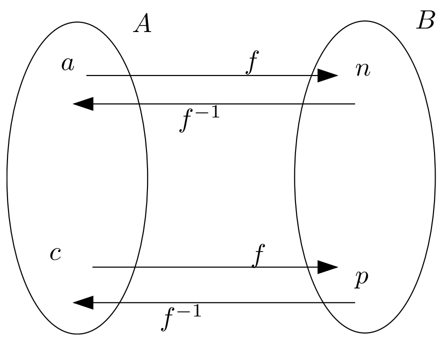

Figure 5.4: If a map and its inverse are (1-) the the map is said to be bijective

Definition 71. The **identity map** I maps every element of any domain D onto itself, that is, onto D again. In symbols,

$$f(a) = a \tag{5.2.9}$$

for every element a of a domain D

Definition 72. Let $A \xrightarrow{f} B$ and let $B \xrightarrow{g} C$ then the **function product** of g and f, also called the **composition** of g and f, and denoted by $g \circ f$ is defined to be the set of all ordered pairs $(a, g(f(a)))$ such that $a \in A$ and $g(f(a)) \in C$.

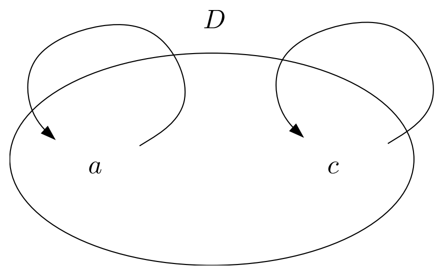

Figure 5.5: The identity map assigns elements of a domain to the same elements

This is equivalent to say that $(a, g(f(a))) \in g \circ f$.

Not every two maps have a product. For the product $g \circ f$ to exist, g must be defined on a set that contains the range of f.

The product of two maps is not always "commutative"; that is, the order in which two maps are written is significant.

5.2.2 Real mappings

Definition 73. A map is **closed** on a set U if the elements of the domain and the elements of the range belong to U. Otherwise the map is **open**. If the set U coincides with the real numbers set, then the map is said to be **real map**

Real mappings are perhaps the most used transformations in applied mathematics. Those are the mappings that produce the curves we are accustomed to see in analytic geometry and calculus when the Cartesian coordinates are used to plot graphics.

For example the quadratic real function is the one that assigns to a real number in a domain another real number that is the square of element of the domain.

The plotting of real functions works under the assumption that the Y-axis is identical with the X-axis, however, under the Transcomplex Number Theory this is not the case. We assume that there is one and only one X-axis, and that the Y-axis is the set of the image-real numbers, that is, the Y-axis is identical only to the set $\mathbb{R}^{\tilde{}}$

If the real quadratic function were to be plotted using the X and $Y^{\tilde{}}$-axes paradigm, the resultant graph will be the following:

5.2.3 Real image maps

In order to view a real function the same way it is viewed using Cartesian coordinates, the following definition is needed.

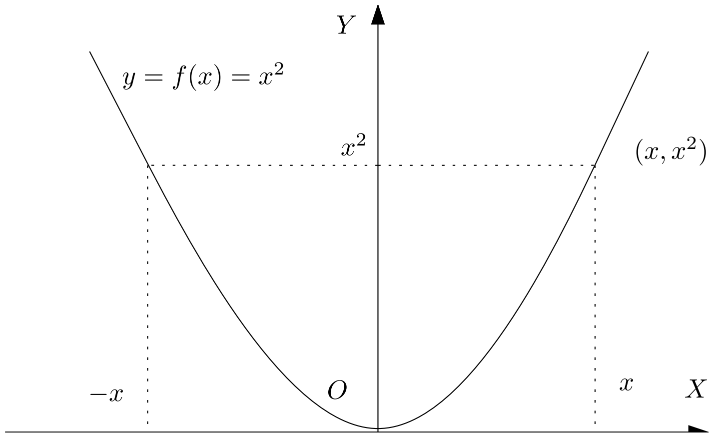

Figure 5.6: The quadratic real function assigns to a real number its square

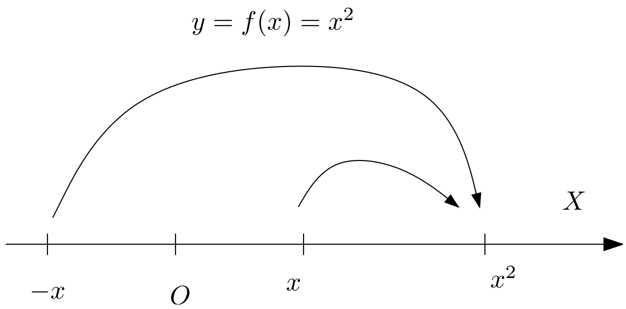

Figure 5.7: Plotting real functions from the X-axis to the X-axis again is the correct way, but yields no useful graph

Definition 74. For a real map f, the **image-map** of f, denoted by the same letter used for the map but with a tilde above, $f^{\sim}$, is the correspondence between the elements of a domain to images of the elements of the domain.

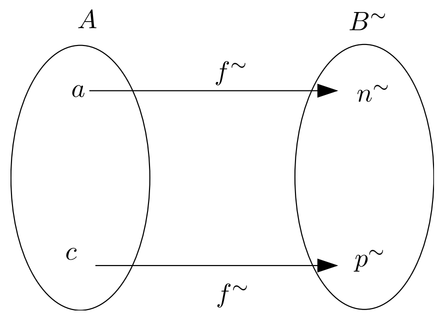

Figure 5.8: The range of a real image-map is always a set of image-real numbers

If the previous real quadratic function (See Fig.) is plotted using the concept of real-image map, the visual result will be the same curve while at the same time consistency in the concepts-usage is preserved.

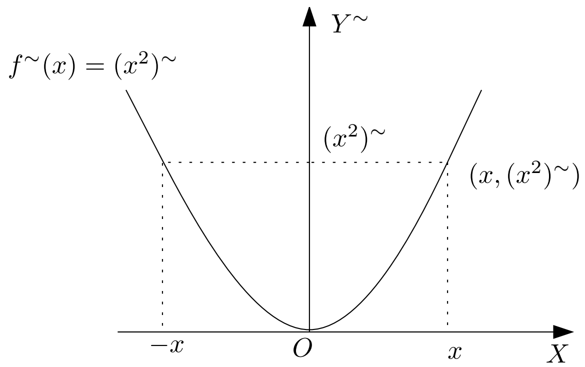

Figure 5.9: The quadratic image-real function assigns to a real number the image of its square

5.2.4 Complex functions

Definition 75. A **complex map**, denoted by capital letters such as F, G, etc., is a map from the complex numbers into itself. That is:

$$F(x + iz) = u + iv \tag{5.2.10}$$

where

$$u = f(x, z) \qquad \text{and} \quad v = g(x, z). \tag{5.2.11}$$

The maps f and g are two independent real functions of the two real variables x and z. Another way to write the above equations is

$$F(x + iz) = f(x, z) + g(x, z)i \tag{5.2.12}$$

or

$$F(x, z) = (f(x, z), g(x, z)i). \tag{5.2.13}$$

Note that in the above equations, $f(x, z) + g(x, z)i$ is just a complex variable where $f(x, z)$ is the real part and $g(x, z)$ is the imaginary part.

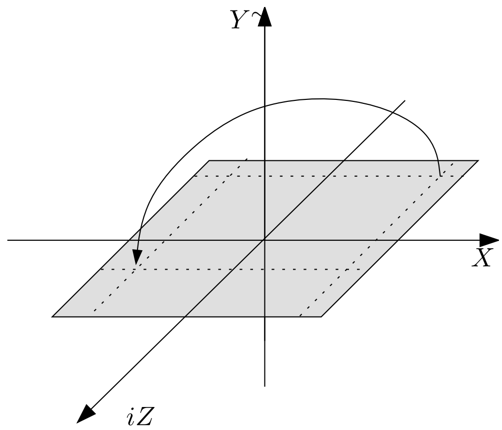

Figure 5.10: A complex map is an assignment of a complex number to another complex number

The current practice in the plotting of complex variable maps is to make two separate complex coordinate planes; one to draw the region of the domain, and the other to draw the region of the range of the map.

The figure below illustrates what is meant in the above paragraph.

Concrete examples of how two complex planes are used to plot complex mappings are found in the study of conformal transformations. Conformal maps are those that preserve angles within regions when transformed under complex maps. One of those maps is the trigonometric sine function

$$W = \sin(Z). \tag{5.2.14}$$

This map transforms a rectangle in the complex Z-plane into a region between two concentric ellipses as shown below.

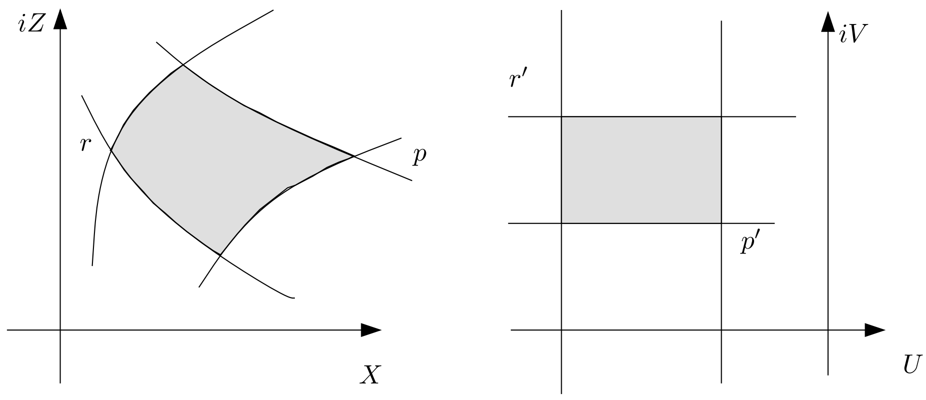

Figure 5.11: Using two complex planes to visualize the behavior of a complex map

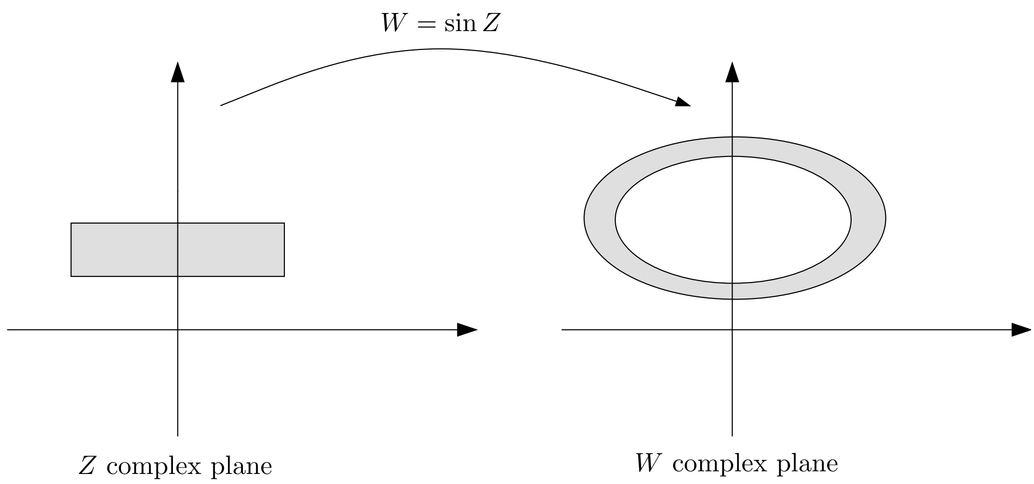

Figure 5.12: The complex trigonometric function $W = \sin Z$ maps a complex rectangular strip into an elliptical region

Details of how this transformation is plotted this way and why a rectangle maps into an elliptical region under the complex trigonometric sine function are found in current complex variables textbooks. This book is concerned about how this transformation is plotted using the concepts here introduced and not about how this is done elsewhere.

Respect to this specific transformation, the reader should be aware that the Z and W planes are ultimately the same planes, and that the only reason they are drawn as separate planes is to give significance to the visual perception of the behavior of the map. However, one of the objectives of the Theory of Transcomlex Numbers is to give significance and cohesion to those two separate complex planar maps and make it possible to plot both -the domain and the range- as a unified entity.

We'll not go into further details here because the next chapter is devoted to the plotting of transcomplex functions and this trigonometric map is among the ones to be covered.

There the reader will find full details and will see how the transcomplex functions not only unifies the Z and W-planes concepts, but will also see that the so-called W-plane plotting is just a proyection of a spatial surface into a plane.

5.3 Transcomplex functions

5.3.1 Semicomplex maps

Definition 76. Let F be a complex map given by

$$F(x + iz) = f(x, z) + g(x, z)i \tag{5.3.1}$$

then the **semicomplex map** of F, denoted by $F^{\tilde{}}$, is a similar and symmetric map to F given by the following rule:

$$F^{\tilde{}}(x, z) = f(x, z)^{\tilde{}} + g(x, z)i. \tag{5.3.2}$$

Below is a picture that shows the behavior of a complex variable under a semicomplex map.

Note that a semicomplex map plots complex numbers from the XiZ plane to corresponding points in the plane $Y^{\tilde{}}iZ$. Then, a semicomplex map is a transformation from the osculating plane $p = (a, 0, c, 0)$ into the rectifying plane $p = (0, b, c, 0)$.

The semicomplex numbers belong to the $Y^{\tilde{}}iZ$ plane. That implies that semicomplex maps are correspondences between complexs and the semicomplexs.

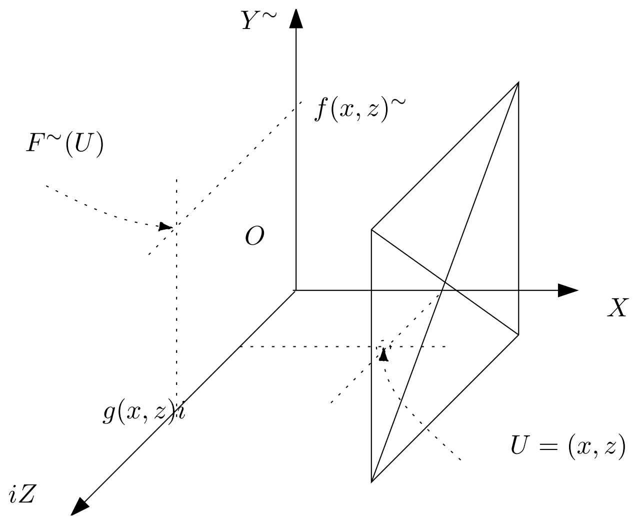

Figure 5.13: A semicomplex map assigns a complex number in the complex plane to another point in the rectifying plane

The net difference between complex mappings and semicomplex maps is that instead of using two separate planes, as shown previously, the semicomplex maps are transformations from one plane of S^4 into another plane of S^4.

In the case of a complex numbers region in the complex plane, the semicomplex map transforms that region into another region in the rectifying plane. The following figure shows that behavior.

So, up to now the progress is not too much because what we have done is only integrating the concept of two separate planes into the unified space S^4. However, the next step will make the big difference.

5.3.2 Transcomplex maps

Definition 77. A **transcomplex map**, denoted by capital letters with an asterisk at its top right, like F^*, is a rule from the complex numbers plane into the real transcomplex numbers space defined by

$$F^*(U) = (Re(U), Re(F(U)), Re(Im(U))) \tag{5.3.3}$$

where F(U) is a complex function U = x + iz.

Note that the definition of transcomplex map is stated in terms of the previous concept of complex map, $F(U)$.

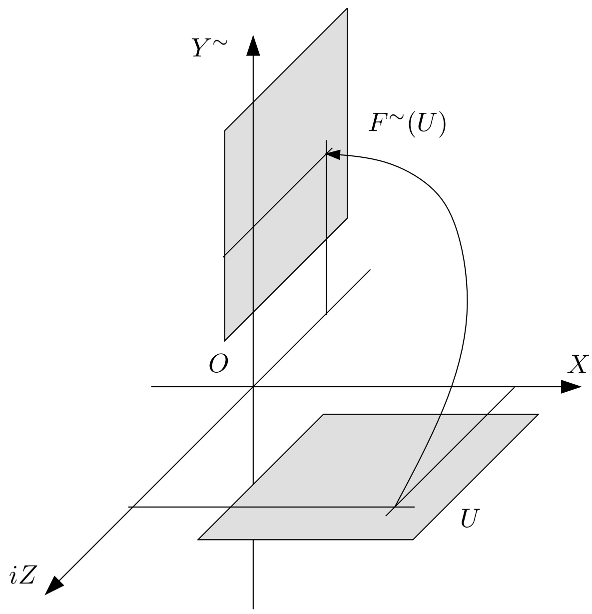

Figure 5.14: A semicomplex map assigns a complex region in the complex plane to a region in the rectifying plane

Since a complex map $F(U)$ can also be stated as a set of two real maps, $f(x, z)$ and $g(x, z)$, the following simple theorem takes charge in stating a transcomplex map in terms of those two real maps.

THEOREM 5.1. *Let F *be a transcomplex map from a domain A of complex variables $U = (x, 0, z)$ to a range B of transcomplex numbers $W = (x', y', z')$. Then*

$$F^*(x, z) = (x, f(x, z), g(x, z)). \tag{5.3.4}$$

Proof. (See the chapter on Theorem Proofs) □

Note that a transcomplex map is a transformation from a plane to a space, that is, from a two-entries ordered pair (the complex numbers) to a three-entries ordered pair (the real transcomplex space).

That is the same to say that transcomplex map are transformations from transcomplex numbers type 10 to transcomplex numbers type 14.

However, since the complex plane is also part of the $XY\tilde{}iZ$ cubicle, we can also say that the transcomplex functions are maps from the tridimensional $XY\tilde{}iZ$ real cubicle into itself.

The following theorem still simplifies the notation of the transcomplex map concept.

THEOREM 5.2. *A transcomplex map $F^*(U)$ is a semicomplex map $F\tilde{}(U)$ plotted in the plane $X = x$. In symbols,*

$$F^*(U) = x + F\tilde{}(U). \tag{5.3.5}$$

Proof. (See the chapter on Theorem Proofs) □

In view of all the concepts stated we can make the following observations:

- Transcomplex maps are semicomplex maps "displaced" along the X-axis.
- The point $x + y\tilde{} + iz$ is the same point $y\tilde{} + iz$ at the plane $X = x$.
- The transcomplex map $F^*(U)$ is the same as the semicomplex map $F^*(U)$ at the plane $X = x$.

Now it is clear the intention behind the introduction of the concept of semicomplex map. The concept of semicomplex map was introduced with the intention of simplifying the plotting of transcomplex functions, which are coming soon.

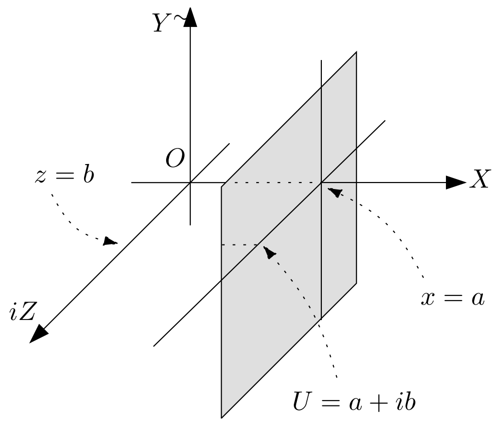

Figure 5.15: Every complex number $U = a + ib$ where $x = a$ is part of the plane $A = a$

Note that now does not apply to say that F^* is plotted"directly above or below" U, as when we plot real variables, that $f(x)$ is plotted above or below x, depending on the sign of $f(x)$.

The analogy still exists, but in terms of planes: now U and $F^*(U)$ are on the same vertical plane. When that plane is seen from an edge, looking toward the normal plane, it appears that U and $F^*(U)$ are on the same vertical line.

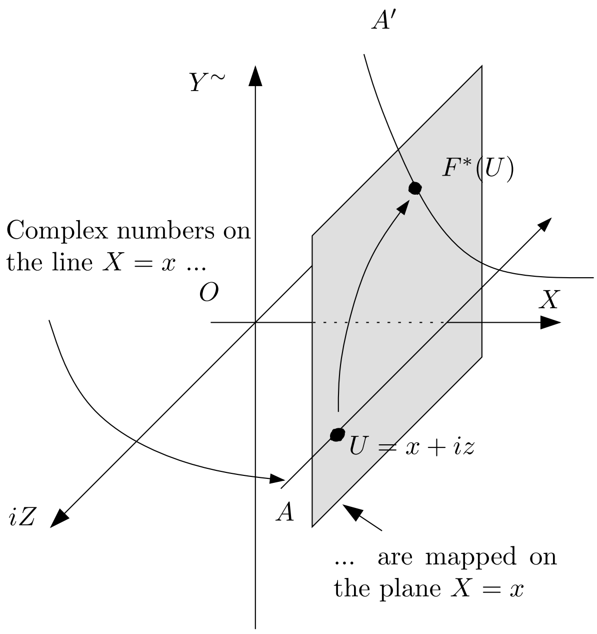

Figure 5.16: The essence of transcomplex maps. The line A' is the line A transformed by the transcomplex map F^*

THEOREM 5.3. *If the domain D of a transcomplex function is of real numbers only, then that transcomplex function reduces to its simple real function expression. That is:*

$$F^*(U) = f(x) \tag{5.3.6}$$

when $x \in \mathbb{R}$.

Proof. (See the chapter on Theorem Proofs) □

5.4 Transcomplex surfaces

5.4.1 The rectifying transformation

By successive application of transcomplex transformations of the parallel lines $X = a$, $X = a'$, $X = a''$, crossing a region of the complex plane, we obtain successive curves $C(a)$, $C(a')$, $C(a'')$,... as mates (that is "images" of the parallel lines) that accumulates one near the other until they generate a surface as shown below.

Fortunately, the curves $C(a)$, $C(a')$, $C(a'')$,... are generated by the same transformation. In view of that fact, we give a name for it:

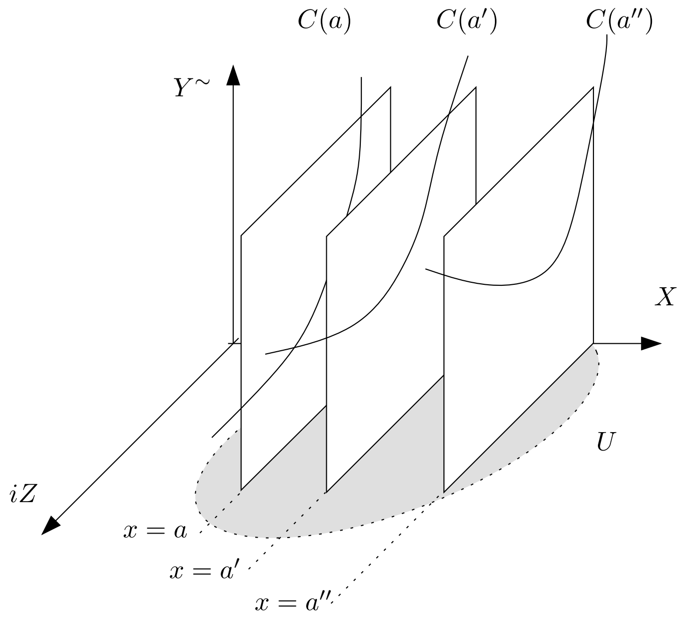

Figure 5.17: The succession of lines $L(a)$, $L(a')$, $L(a'')$ generate the succession of curves $C(a)$, $C(a')$, $Ca'')$ and the curves eventually produce a surface

Definition 78. The transformation obtained at the intersection of the rectifying plane $X = x$ with a transcomplex surface will be called the **rectifying transformation** at $X = x$.

A rectifying transformation at $X = x$ is generated precisely by the set of parametric equations of the complex map of the transcomplex map. Thus in the figure, if the transcomplex map is

$$F^*(x,z) = (x, f(x,z), g(x,z)) \tag{5.4.1}$$

then a rectifying transformation is the transformation resulting from the set of real equations

$$y\tilde{} = f(x,z) \qquad \text{and} \qquad z = g(x,z). \tag{5.4.2}$$

5.4.2 Transcomplex surfaces

It was mentioned above that a succession of rectifying transformations produce a surface.

Definition 79. The set of all curves $C1$, $C2$, $C3$,... obtained by the rectifying transformations at the planes $X = a$, $X = a'$, $X = a''$,... by the transcomplex map $F^*(U)$ is called the **transcomplex surface** of $F^*(U)$.

Transcomplex surfaces are a special kind of surface; so special that it is unique of the Transcomplex Number Theory. In fact, the surfaces generated by transcomplex maps are not found anywhere, in any textbook, reference, or workbook. This is so because they are plotted using a new paradigm of coordinates: the space S^4.

5.4.3 The normal transformation

In the same that a transcomplex surface can be intercepted by the rectifying plane $X = x$, it can be also intercepted by any other plane. Of special interest is the interception with the normal plane $Z = 0$.

Definition 80. The transformation obtained at the intersection of the normal plane $Z = z$ with a transcomplex surface will be called the **normal transformation** at $Z = z$.

If the normal plane is set at the point $Z = 0$, then we are dealing with common "Cartesian" plane, or the so-called "real plane".

In the Introduction of this book it was said:

> ...complex functions need not have resemblance with the original real function graph, but intuition tell us that in the same way that the real numbers come out when the imaginary part of the complex numbers are chosen to be zero, so the graphs of the real functions should also come out when it is equated to zero the imaginary axis of the complex graph of a complex function.

Now we see that that our intuition was right because it is possible to obtain real graphs out of complex graphs. What was "wrong" was the classical approach toward the plotting of complex functions. Plotting functions using two separate complex planes is the barrier or wall that does not permit us to visualize a complex map a tridimensional surface.

THEOREM 5.4. *Let $U = (x, z)$ be a complex region on the complex plane and let $F(U) = f(x, z) + g(x, z)i$ be the complex function defined on the region U. Then the interception of the transcomplex map $F^*(U)$ with the XY˜plane is the same as the image-real map $f(x)$ defined for the subdomain of U of all ordered pair such that $z = 0$.*

Proof. (See the chapter on Theorem Proofs) □

Chapter 6

Transcomplex Surfaces

6.1 Introduction

In the introduction of this book, it was stated that the purpose of it is to demonstrate that rather than using the usual two separate complex planes mapping, there is one more pleasant and coherent alternative to plot functions of complex variables. The transcomplex mapping system offers that alternative, because it plots on a 3-dimensional space to a 3-dimensional space basis. And that makes a lot of difference as will be seen in this chapter.

But this new way of plotting complex variables is not the product of an arbitrary mental creation. So far, throughout the preceding chapters, every effort was directed to show that the transcomplex numbers are completely justified. All this was necessary because this chapter–devoted to the plotting of transcomplex functions–will come out with a couple of surprises.

One of our goals is to show that the common real graphs of real functions found in mathematics books are algebraically and geometrically special cases of complex functions. That real functions are special cases of complex function is nothing new at all, but the problem arises with the graphical representation of both, because there seems to be an abyss between them.

We have already seen that in the 3-dimensional space the $XY\tilde{\ }iZ$ complex functions are special cases of transcomplex functions, which in turn represent the most general cases of hypersurfaces.

We have also seen that the domain and range of a function can be any transcomplex number set, and that if the domain is the complex numbers class, then that subset is located in the horizontal XiZ-plane. The "image" under any map of a point on that plane, let's say the point

$$(x, 0, z, 0) = x + iz,$$

is the real transcomplex number

$$\Big(x, Re\big(f(x+iz)\big), Re\Big(Im\big(f(x+iz)\big)\Big)\Big).$$

Finally, we know that every point, be it real, complex, or any other, have coordinates in the four-coordinates simultaneously. Thus, the true graph of a function should be viewed using the 4 subcubicles of S^4. Every cubicle generates its own surface for every complex map. Those 4 surfaces —when mentally combined— make the hypersurface that we will try to grasp or comprehend in our minds.

In this chapter it will be appreciated the true beauty of the complex numbers and of the space S^4, and in particular, the real subspace S^3. Simple real continuous functions, like the quadratic equation, will transform into an unimaginable curved sheet in space (in fact, into a horse saddle-like curve). The simple logarithmic curve will come out to be an infinite set of semi black holes linked by one thin line, a direct consequence of the map of logarithms of negative numbers.

A note about the graphics that follows: The previous illustrations of this book were done with software that can produce high quality postscript output.

On the contrary, the illustrations that follows from this point on are low resolution screen captures that had been resized, cropped, and converted to a gray scale.

The original graphics were done with 4DLab, a special program I coded for the illustrations of this chapter.

6.2 Methodology

Respect to the variables used, U and W will respectively denote the domain and the range of the transcomplex function under discussion

$$W = F^*(U). \tag{6.2.1}$$

Although a complex variable can be represented as $U = x + iz$, we will give some preference to the ordered pair notation: $U = (x, 0, z)$. In the same way: $W = x + y\tilde{} + iz$ will be written also as $W = (x, y, z)$.

In order to avoid confusion, while working with the mappings, specially with parametric equations, sometimes we will temporarily use other symbols for the variables of W, as follows:

$$W = (x', y'z'). \tag{6.2.2}$$

For every case that follows we will discuss the behavior of the function in the real cubicle only. Thus, the elements of the domain U are always complex points of the form

$$U = (x, 0, z) \tag{6.2.3}$$

and the elements of the range W are transcomplex points of the form

$$W = (x', y'z') = F^*(x, 0, z). \tag{6.2.4}$$

In accordance with what has been previously discussed, for a given complex function $F(U)$ recall that $F^*(U)$ denotes the transcomplex version of F, while $f(x)$ denotes the real version of the function.

To review from the past chapter:

- **Complex maps** (Eq. 5.2.13) are of the form:

$$F(x, 0, z) = \big(f(x, 0, z), 0, g(x, 0, z)\big). \tag{6.2.5}$$

  This is a transformation of a planar domain of the complex plane into another planar region —the range— of the complex plane again (Fig. 5.10).

- **Semicomplex maps** (Eq. 5.3.2) are of the form:

$$F^{\tilde{}}(x, 0, z) = \big(0, f(x, 0, z), g(x, 0, z)\big). \tag{6.2.6}$$

  This is a transformation of a planar domain of the complex plane into another "vertical" planar region —the range— of the complex plane again on the rectifying plane $X = 0$ (Fig. 5.13).

- **Transcomplex maps** (Eq. 5.3.4) are of the form:

$$F^*(x, 0, z) = (x, f(x, z), g(x, z). \tag{6.2.7}$$

  This transformation puts the elements of a domain of the complex plane into a rectifying plane that crosses the domain at exactly the point X = x (Fig. 5.16). Contrary to the complex map, this is equivalent to say the "image" of a complex number $U = x+iz$ under a complex transformation $F(U)$ is put "above" the variable U instead of placing it on another complex plane. Compare to Fig 5.10.

- **Rectifying maps** (Eq. 5.4.2) are obtained by the pair of equations:

$$y^{\sim} = f(x,0,z) \qquad \text{and} \qquad z = g(x,0,z) \qquad \text{at} \qquad X = x. \tag{6.2.8}$$

- **Real maps** are of the form:

$$y^{\sim} = f(x) \tag{6.2.9}$$

  or

$$f(x) = F^{*}(x,0,0). \tag{6.2.10}$$

For the purpose of uniformity in the discussion of the functions to be presented, all of them go through the following steps:

1. **Preliminaries**

   A short introduction to the function.

2. **The complex version of the function**

   Transcomplex maps are derived from complex maps, so we begin by stating the complex transformation first, since we are familiar with them.

3. **The parametric equations of the complex function**

   Complex maps are decomposed as two parametric equations. The two equations are stated.

4. **The semicomplex map**

   Semicomplex maps are projections of a transcomplex surface into the $Y^{\sim}iZ$ plane.

5. **The transcomplex map**

   Transcomplex maps are the spatial version of complex maps.

6. **The rectifying transformation**

   The rectifying transformation is the key map for plotting transcomplex maps.

7. **The real map**

   The real map is the one obtained with the real subdomain of a complex domain.

Below is a short summary of the formulas only.

Complex maps:	$F(x, 0, z) = (f(x, 0, z), 0, g(x, 0, z))$
Semicomplex maps:	$F(x, 0, z) = (0, f(x, 0, z), g(x, 0, z))$
Transcomplex maps:	$F^{*}(x, 0, z) = (x, f(x, 0, z), g(x, 0, z))$
Rectifying maps:	$y^{\sim} = f(x, 0, z)$ and $z = g(x, 0, z)$ at $X = x$
Real maps:	$\mathrm{y}^{\sim} = f(x)$

Figure 6.1: Summary of the maps and their formulas.

6.3 The Transcomplex Identity Function $F^*(U) = U$

1. **Preliminary**

   The identity function is one of the simplest transcomplex maps. The purpose of this map is to plot an exact image of the points of the domain.

2. **The complex version of $F^*(U) = U$.** The complex identity function is written as:

$$F(x, 0, z) = (x, 0, z). \tag{6.3.11}$$

3. **The parametric equations of $F^*(U) = U$.** Reviewing the definition of complex map (Eq. 5.2.13)

$$F(x, 0, z) = (f(x, 0, z), 0, g(x, 0, 0)) \tag{6.3.12}$$

   and applying this definition to the identity function above, we have

$$(x, 0, z) = (f(x, 0, z), 0, g(x, 0, 0)). \tag{6.3.13}$$

   Equating the above ordered pair's components we obtain:

$$f(x, 0, z) = x \qquad \text{and} \qquad g(x, 0, z) = z. \tag{6.3.14}$$

4. **The semicomplex map**

   Using the definition of semicomplex maps (Eq. 5.3.2) and replacing $f(x, 0, z)$ by x and $g(x, 0, z)$ by z (the previously obtained results) we have:

$$F^{\tilde{}}(x, 0, z) = (0, x, z). \tag{6.3.15}$$

   By simple visual inspection it can be noted that the semicomplex map merely consists of an exact portrait of the complex plane, but projected on the $iZY^{\tilde{}}$-plane.

   The fact that the first entry of the ordered pair $(0, x, z)$ is zero means that everything occurs in the $iZY^{\tilde{}}$-plane where the x-component is zero.

5. **The transcomplex map** Doing the same replacements of $f(x, 0, z)$ and $g(x, 0, z)$ by x and z respectively on the definition of transcomplex map we obtain:

$$F * (x, 0, z) = (x, x, z). \tag{6.3.16}$$

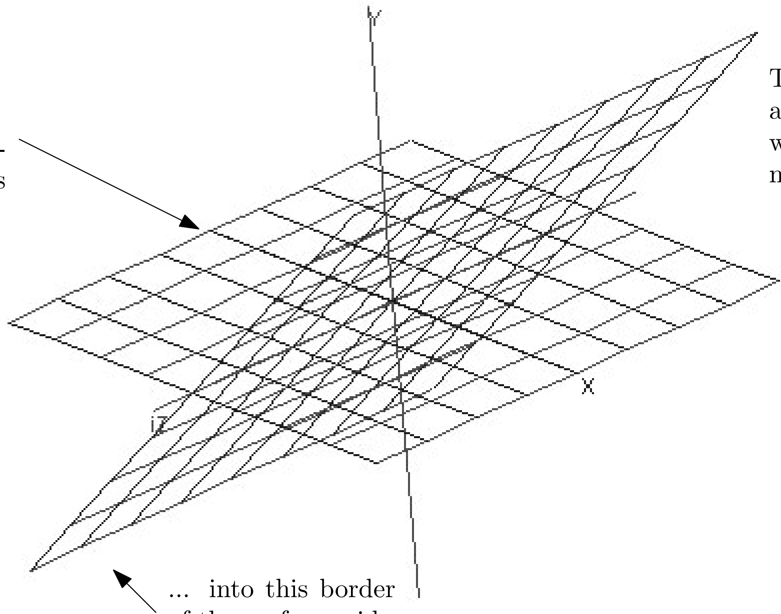

Figure 6.2: Under the identity transcomplex map, the surface generated is also plane, makes 45 degrees with the complex plane, but it is stretched along the real component by a factor of $\sqrt{2}$.

6. **The rectifying transformation**

   We have that for the unit function the rectifying map becomes:

$$y^{\sim} = f(x, 0, z) = x \qquad \text{and} \qquad z = g(x, 0, z) = z \tag{6.3.17}$$

   or simply,

$$y^{\sim} = x^{\sim} \qquad \text{and} \qquad z = z. \tag{6.3.18}$$

   We can be deduced that at the point $x = a = const$, any line parallel to the iZ-axis is plotted at a height $y^{\sim} = a^{\sim}$on the $iZY^{\sim}$-plane. The length of the line plotted on the $iZY^{\sim}$-plane is equal to the length of line of the domain parallel to the iZ-axis.

7. **The real map**

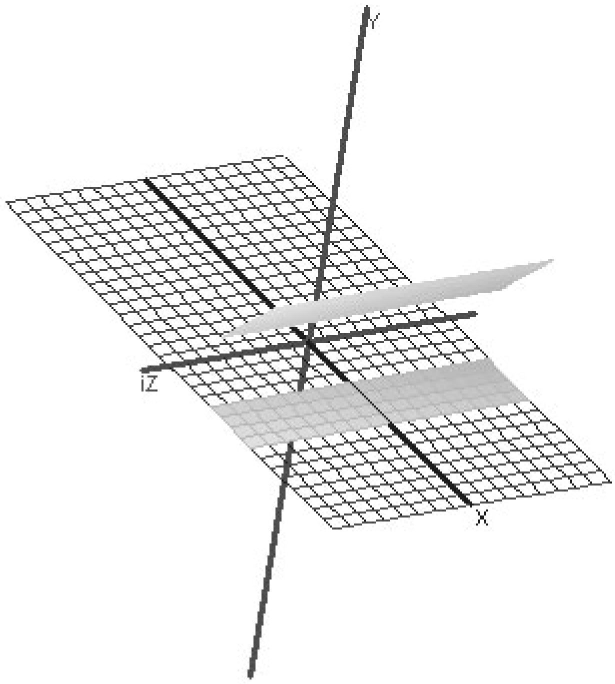

Figure 6.3: The transformation of a thin strip that runs parallel to the iZ-axis.

The real map is found when we set $z = 0$ in the domain of the transcomplex map. Thus,

$$f(x) = F * (x, 0, 0) = (x, x, 0). \qquad (6.3.19)$$

In the common usage of the Cartesian coordinates, this is written as:

$$y = f(x) = x. \qquad (6.3.20)$$

Since the transcomplex surface of the unit function is an inclined plane, when it intercepts the $XY^{\sim}$-plane a single inclined line with a slope of 45 degrees is obtained.

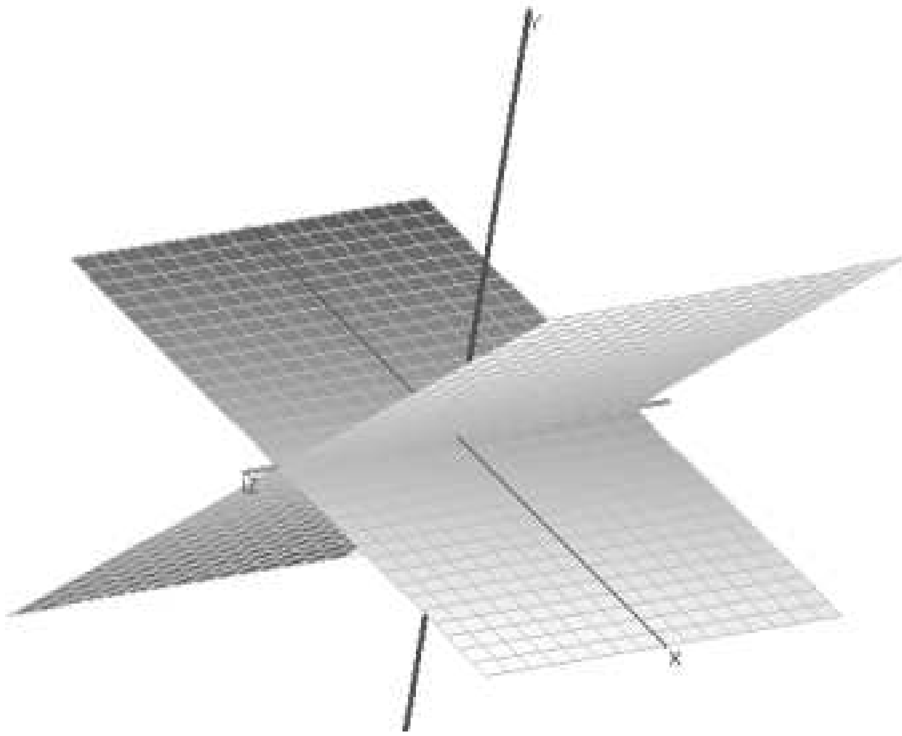

Figure 6.4: The transcomplex identity surface, when looking toward the origin of coordinates. At the plane $iZ = 0$, the surface becomes a diagonal line, which is exactly the real function $y = x$.

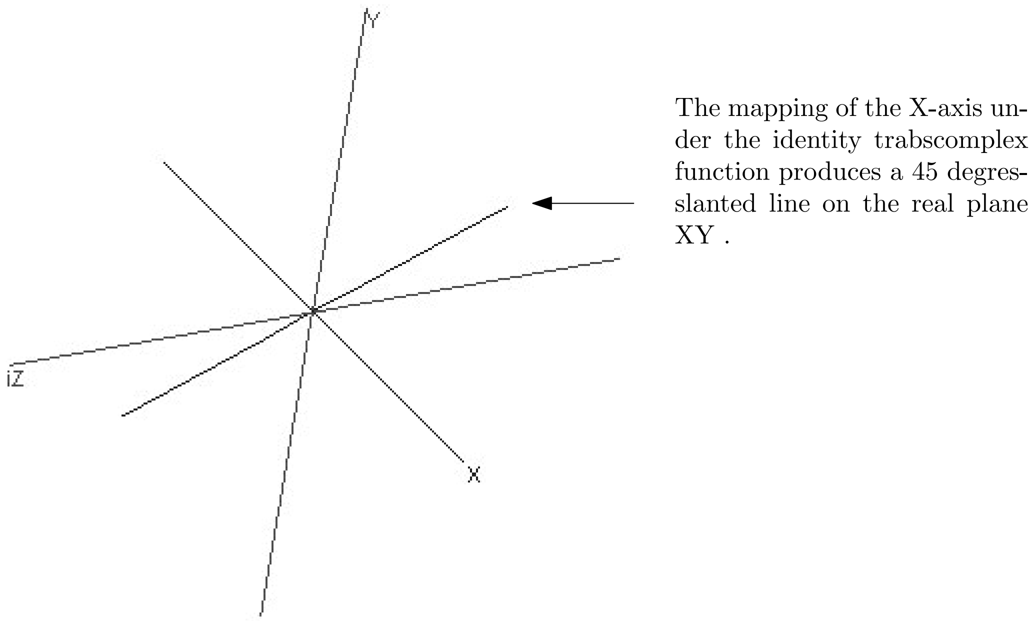

Figure 6.5: The simple $y = x$ function we usually plot on the Cartesian plane is just a "section" of the planar spatial surface $F*(U) = U$.

6.4 The Transcomplex Quadratic Function $F^*(U) = U^2$

1. **Preliminary**

   The quadratic function is also one of the simplest maps. The purpose of this map is to plot an image of the square of the points of the domain.

2. **The complex version of $F^*(U) = U^2$.** The complex quadratic function is written as:

$$F(x, 0, z) = (xx - zz, 0, xz + zx) = (x^2 - z^2, 0, 2xz). \tag{6.4.21}$$

3. **The parametric equations of $F^*(U) = U^2$.** Reviewing the definition of complex map (Eq. 5.2.13)

$$F(x, 0, z) = (f(x, 0, z), 0, g(x, 0, 0)) \tag{6.4.22}$$

   and applying this definition to the identity function above, we have

$$(x, 0, z) = (f(x, 0, z), 0, g(x, 0, 0)). \tag{6.4.23}$$

   Equating the above ordered pair's components we obtain:

$$f(x, 0, z) = x^2 - z^2 \qquad \text{and} \qquad g(x, 0, z) = 2xz. \tag{6.4.24}$$

4. **The semicomplex map**

   Using the definition of semicomplex maps (Eq. 5.3.2) and replacing $f(x, 0, z)$ by $x^2 - z^2$ and $g(x, 0, z)$ by $2xz$ (the previously obtained results) we have:

$$F^{\tilde{}}(x, 0, z) = (0, x^2 - z^2, 2xz). \tag{6.4.25}$$

   By visual inspection it can be noted that the semicomplex map merely consists of an square of the complex plane, but projected on the $iZY^{\tilde{}}$-plane.

   The fact that the first entry of the ordered pair $(0, x, z)$ is zero means that everything occurs in the $iZY^{\tilde{}}$-plane where the x-component is zero.

5. **The transcomplex map** Doing the same replacements of $f(x, 0, z)$ and $g(x, 0, z)$ by x^2 and $2xz$ respectively on the definition of transcomplex map we obtain:

$$F * (x, 0, z) = (x, x^2 - z^2, 2xz). \tag{6.4.26}$$

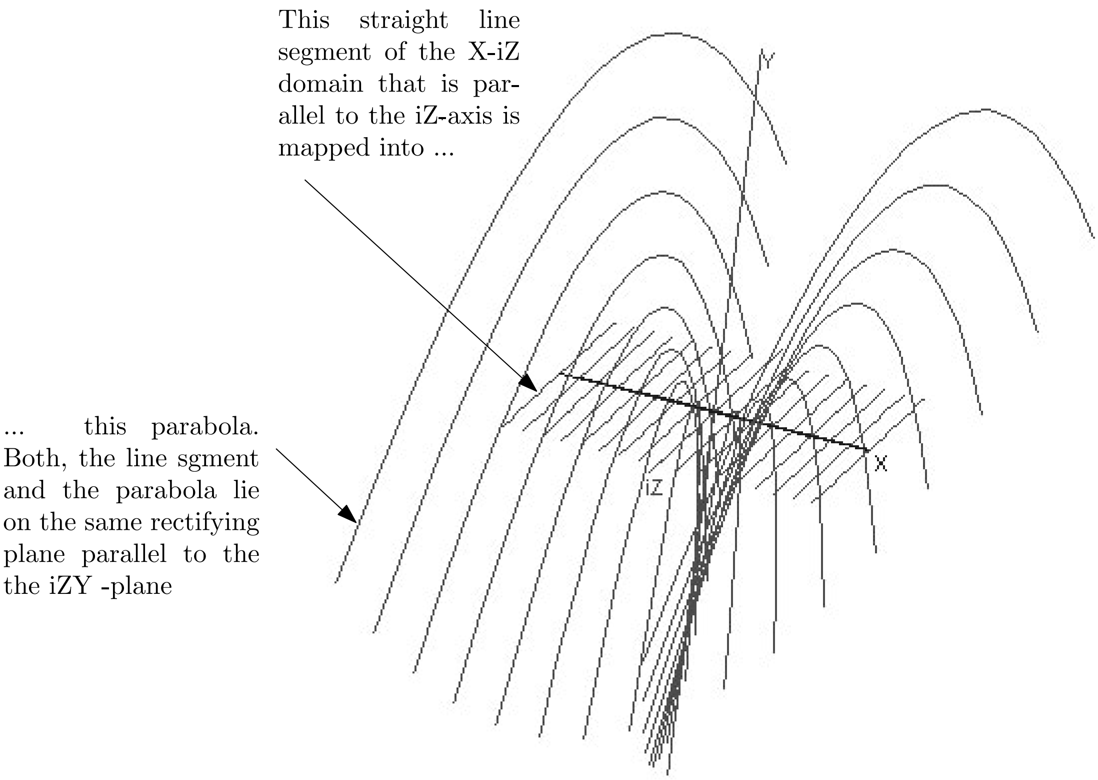

Figure 6.6: A good example of a series of semicomplex maps. Starting at the negative end of the X-axis, we see a series of parabolas, each one corresponding to the semicomplex map of line segments parallel to the iZ-axis under the transcomplex function $F(U) = U^2$.

We also have that for the quadratic function the rectifying map becomes:

$$y^{\sim} = f(x, 0, z) = x^2 - z^2 \qquad \text{and} \qquad z = g(x, 0, z) = 2xz \tag{6.4.27}$$

or simply,

$$y^{\sim} = (x^2 - z^2)^{\sim} \qquad \text{and} \qquad z = 2xz. \tag{6.4.28}$$

By the pair it can be deduced that at the point $x = a = const$, any line parallel to the iZ-axis is plotted as $y^{\sim} = (x^2 - z^2)^{\sim}$on the $iZY^{\sim}$-plane.

But $(x^2 - z^2)^{\sim}$is the equation of a parabola on a plane X = x, a plane parallel to the $iZY^{\sim}$-plane. So, the surface of the function $F^*(U) = U^2$ is made-up of a sequence of parabolas. The surface is symmetric respect to the $iZY^{\sim}$-plane and always opens toward the negative side of the $Y^{\sim}$-axis.

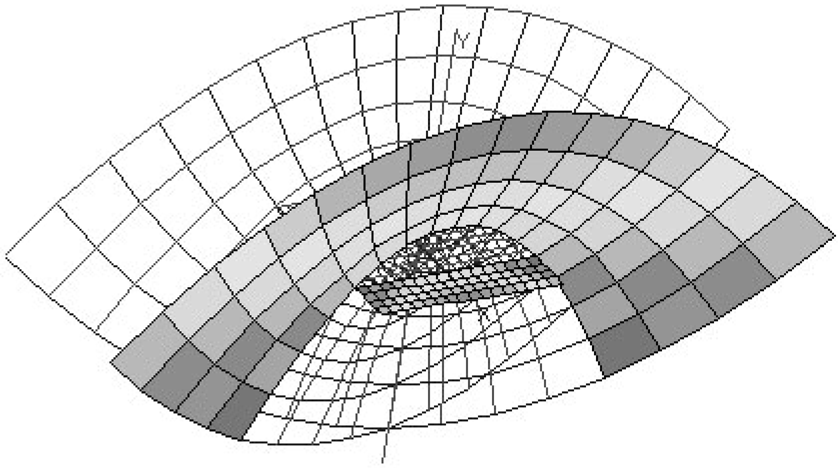

Figure 6.7: The surface generated by a thin strip of a rectangular domain. This strip runs parallel to the iZ-axis. The curved surface is all made-up of a sequence of contiguous parabolas.

These equations are parametric equations (in terms of the parameter z) of the parabola $z^2 = 4q^2(q^2 - y)$.

6. **The real map**

   The real map is found when we set $z = 0$ in the domain of the transcomplex map. Thus,

$$f(x) = F * (x, 0, 0) = (x, x^2 - z^2, 0) = (x, x^2, 0). \tag{6.4.29}$$

   In the common usage of the Cartesian coordinates, this is written as:

$$y = f(x) = x^2. \tag{6.4.30}$$

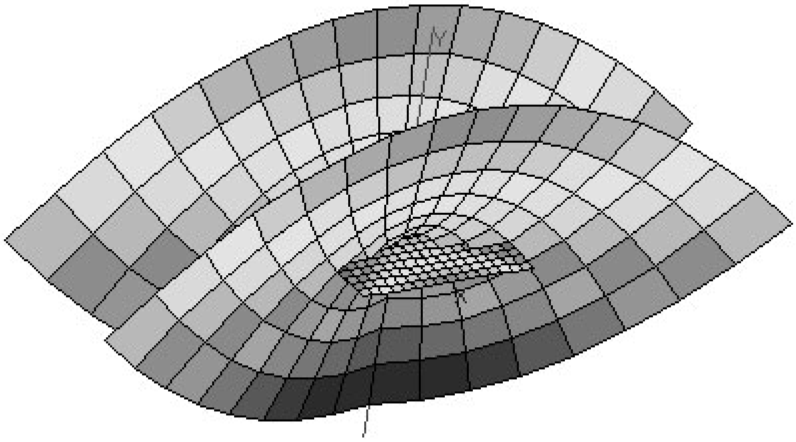

Figure 6.8: The transcomplex quadratic surface, when looking toward the origin of coordinates. At the plane $iZ = 0$, the surface becomes the usual parabola, which is exactly the real function $y = x^2$.

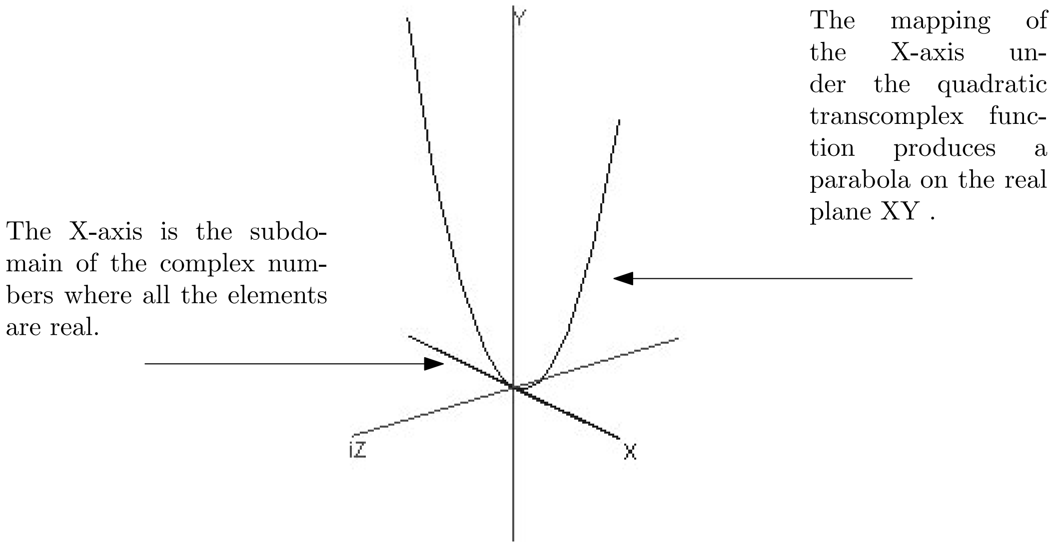

Figure 6.9: The real parabola we see under the real map $y = x^2$ is only a "section" of a more general spatial surface. That surface is the one generated under the transcomplex map $F(U) = U^2$.

Chapter 7

Theorem Proofs

7.1 Theorems from Chapter 1: Ordered Pairs

Section: **Addition and multiplication of ordered pairs** (1.4 Page 10 [1])

THEOREM 1.1. *Let a and b be any two unknown numbers belonging to any of the X or $Y\tilde{}$-axes, then $ab = 0$ if only if a belongs to X and b belongs to $Y\tilde{}$ or the converse. In symbols:*

$$ab = 0$$

if and only if

$$a \in X \qquad \textit{and} \qquad b \in Y\tilde{}$$

or

$$a \in Y\tilde{} \qquad \textit{and} \qquad b \in X$$

where

$$X = \mathbb{R} \qquad \textit{and} \qquad Y\tilde{} = \mathbb{R}\tilde{}.$$

Proof. Suppose that a is a real number and that b is an image-real number; i.e.,

$$a \in X \qquad \text{and} \qquad b \in Y\tilde{}.$$

Then their product is zero. If we assume the contrary for a and b:

$$a \in Y\tilde{} \qquad \text{and} \qquad b \in X$$

their product will be also zero.

[1]Ref. Sec. 1.4

Now suppose that a and b are real, $ab = 0$ if and only if at least one of the factors is zero. Suppose that $a \neq 0$, then $b = 0$, but since both are real, both belong to X; however, since $b = 0$, it is true that b belongs to $Y\tilde{}$. If we assume the contrary, that is, that $b \neq 0$ then $a = 0$ and in this case b belongs to X and a belongs to $Y\tilde{}$.

If both are image reals, then both belong to $Y\tilde{}$. If $a \neq 0$ then $b = 0$, but since both are image-reals and $b \neq 0$, it is true that b belongs to X. If we assume the contrary, that $b \neq 0$, then $a = 0$ and in this case b belongs to $Y\tilde{}$ and a belongs to X.

If both are zero, it is no matter true that

$$0 \in X \qquad \text{and} \qquad 0 \in Y$$

or

$$0 \in Y \qquad \text{and} \qquad 0 \in X.$$

The converse is alto true, that if

$$0 \in X \qquad \text{and} \qquad 0 \in Y\tilde{}$$

or

$$0 \in Y\tilde{} \qquad \text{and} \qquad 0 \in X$$

then

$$0 * 0 = 0 \qquad \text{and} \qquad 0 * 0 = 0$$

is obviously true. □

Section: **The ordered pairs field** (1.6 Page 20 [2])

THEOREM 1.2. *The class $\mathbb{O}$, that is, the class of all real numbers ordered pairs, together with the operations + and *, make a field.*

Proof. To prove that the class $\mathbb{O}$ and the operations + and * make a field, we have to go trough all the twelve requisites imposed in the previous definition of the concept of field.

Let

$$A = (a, b), \qquad B = (a', b')\text{and} \qquad C = (a'', b'')$$

be any three elements of $\mathbb{O}$.

1. **Closure** of + in $\mathbb{O}$.

$$A + B = (a + a', b + b')$$

[2] Ref. Sec. 1.6

but

$$a + a' \qquad \text{and} \qquad b + b'$$

are both reals, therefore $A + B$ always exist.

2. **Associativity** of $+$ in $\mathbb{O}$.

$$A + (B + C) = \big(a + (a' + a''), b + (b' + b'')\big)$$

but,

$$\big(a + (a' + a''), b + (b' + b'')\big) = \big((a + a') + a'', (b + b') + b''\big)$$

because the reals are associative under addition. Therefore,

$$A + (B + C) = (A + B) + C.$$

3. **Commutativity** of $+$ in $\mathbb{O}$.

Since $a + a' = a' + a$ and $b + b' = b' + b$ by the commutativity of addition of reals, then $A + B = B + A$.

4. The **neutral element** of $+$ in the class $\mathbb{O}$ is the pair $(0, 0) = 0$ because

$$\begin{aligned} A + 0 &= (a, b) + (0, 0) \\ &= (a + 0, b + 0) \\ &= (a, b) \\ &= A. \end{aligned}$$

5. The **negative element** of $+$ in the class $\mathbb{O}$ is the ordered pair $-A = (-a, -b)$ of $\mathbb{O}$ because

$$\begin{aligned} A + (-A) &= (a, b) + (-a, -b) \\ &= (0, 0) \\ &= 0. \end{aligned}$$

6. **Closure** of $*$ in $\mathbb{O}$.

$$A * B = (a * b, a' * b')$$

but $a * a'$ and $b * b'$ are both real, therefore $A * B$ always exist.

7. **Associativity** of $*$ in $\mathbb{O}$.

$$\begin{aligned} A * (B * C) &= \big(a * (a' * a''), b * (a' * b'')\big) \\ &= \big((a * a') * a'', (b * b') * b''\big) \end{aligned}$$

because the reals are associative under multiplication. Therefore

$$A * (B * C) = (A * B) * C.$$

8. **Commutativity** of * in $\mathbb{O}$.

Since $a * a' = a' * a$ and $b * b' = b' * b$ by the commutativity of multiplication of reals. Then $A * B = B * A$.

9. The **neutral element** of * in the class $\mathbb{O}$ is the pair $(1, 1)$ because

$$\begin{aligned} A * (1, 1) &= (a, b) * (1, 1) \\ &= (a * 1, b * 1) \\ &= (a, b) \\ &= A. \end{aligned}$$

10. The **inverse** of an ordered pair $A = (a, b) \neq (0, 0)$ in the class $\mathbb{O}$ is the ordered pair $A^{-1} = (a^{-1}, b^{-1})$ because

$$\begin{aligned} A * A^{-1} &= (a * a^{-1}, b * b^{-1}) \\ &= \left(\frac{a}{a}, \frac{b}{b}\right) \\ &= (1, 1). \end{aligned}$$

11. **Distributivity of multiplication** in the class $\mathbb{O}$. The multiplication operation is distributive respect to addition in the class $\mathbb{O}$ because

$$\begin{aligned} A * (B + C) &= (a, b) * \big((a', b') + (a'', b'')\big) \\ &= (a, b) * (a' + a'', b' + b'') \\ &= \big(a * (a' + a''), b * (b' + b'')\big) \\ &= (a * a' + a * a'', b * b' + b * b'') \\ &= (a * a', b * b') + (a * a'', b * b'') \\ &= (a, b) * (a', b') + (a, b) * (a'', b'') \\ &= A * B + A * C. \end{aligned}$$

12. If $A * B = 0$ then $A = 0$ or $B = 0$. If $A * B = 0$, then $(aa', bb') = 0$, which implies $aa' = 0$ and $bb' = 0$. But this also implies that

$$a = 0 \qquad \text{or} \qquad a' = 0$$

and at the same time

$$b = 0 \qquad \text{or} \qquad b' = 0.$$

Apart from the trivial cases $A = 0$ or $B = 0$, that means we have the following two possibilities of having a product equal to zero.

$$(a, 0) * (0, b) = 0 = a * (b')$$

and

$$(0, a) * (b, 0) = 0 = a' * (b).$$

But, those multiplications are the commuted case product of a real number times an image real number, two perpendicular numbers for which was proven that their product is always zero.

□

7.2 Theorems from Chapter 2: Complex Numbers

Section: **Complex numbers** (2.5 Page 37 [3])

THEOREM 2.1. *The unit element under multiplication of the complex numbers field is the complex number* $1 + 0i = 1$. *That is, for every complex number* C*:*

$$C * 1 = C.$$

Proof. Let C and C' be any two complex numbers where $C = a+bi = (a, bi)$ and $C' = a'+b'i$ such that

$$C * C' = C.$$

Then

$$\begin{aligned} C * C' &= (a, ib) * (a', ib') \\ &= (aa' - bb', ab' + a'b). \end{aligned}$$

But, if $C * C' = C$, then

$$(aa' - bb', ab' + a'b) = (a, b).$$

Consequently

$$aa' = a \qquad \text{and} \qquad ab' + a'b = b.$$

Solving algebraically for a' and b' we get

$$a' = 1 \qquad \text{and} \qquad b' = 0.$$

Therefore,

$$C' = (1, 0) = 1.$$

□

Section: **Complex numbers** (2.5 Page 37 [4])

THEOREM 2.2. *The inverse of the complex number* $C = a + ci = (a, ci)$ *is the complex number* C^{-1} *given by:*

$$\begin{aligned} C^{-1} &= \frac{a}{a^2 + c^2} - \frac{c}{a^2 + c^2} i \\ &= \Big(\frac{a}{a^2 + c^2}, -\frac{c}{a^2 + c^2} i\Big). \end{aligned}$$

[3]Ref. Sec. 2.5
[4]Ref. Sec. 2.5

Proof. Let C' = a' + bi = (a, bi) be the inverse of C, then

$$C * C' = (1, 0i).$$

But

$$C * C' = \big(aa' - bb', (ab' + a'b)i\big) = (1, 0i).$$

Hence

$$aa' - bb' = 1 \qquad \text{and} \qquad ab' + a'b = 0.$$

From this we get that

$$a' = \frac{a}{a^2 + b^2} \qquad \text{and} \qquad b' = \frac{b}{a^2 + b^2}.$$

Therefore,

$$\begin{aligned} C^{-1} &= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \\ &= \Big(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2} i\Big). \end{aligned}$$

□

Section: **Complex numbers** (2.5 Page 38 [5])

THEOREM 2.3. *Let $C = a + ci$ and $C' = a' + c'i$ be any two complex numbers, then*

$$|C| * |C'| \neq |C * C'|$$

but on the contrary,

$$\|C\| * \|C'\| = \|C * C'\|.$$

Proof. For the first part of the theorem we have:

$$\begin{aligned} |C| * |C'| &= |(a, ib,)| * |(a', ib)| \\ &= (|a|, i|b|) * (|a'|, i|b'|) \\ &= \big(|a| * |a'| - |b| * |b'|, i(|a| * |b'|) + |a'| * |b|\big) \\ &= \big(|aa'| - |bb'|, i(|ab'| + |a'b|)\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} |C * C'| &= \big|\big(aa' - bb', i(ab' + a'b)\big)\big| \\ &= (|aa' - bb'|, i|ab' - a'b|). \end{aligned}$$

[5]Ref. Sec. 2.5

Only for some very special cases the equality

$$|aa'| - |bb'| = |aa' - bb'|$$

and the equality

$$|ab' + a'b| = |aa'| + |a'b|$$

are true concurrently. So that, in general,

$$|C| * |C'| \neq |C * C'|.$$

The second part of the theorem, the one that states that the product of two norms is equal to the norm of the product is easier to demonstrate.

$$\begin{aligned} \| C * C' \| &= \big|(a, ib) * (a', ib')\big| \\ &= \Big|\big(aa' - bb', i(ab' + a'b)\big)\Big| \\ &= \big|\sqrt{(aa' - bb')^2 + (ab' + a'b)^2}\big|. \end{aligned}$$

Expanding the terms in the square root we have:

$$\begin{aligned} \| C * C' \| &= \big|\sqrt{(aa' - bb')^2 + (ab' + a'b)^2}\big| \\ &= \big|\sqrt{(aa')^2 + (bb')^2 + (ab')^2 + (a'b)^2}\big| \\ &= \big|\sqrt{(a^2 + b^2) * (a'^2 + b'^2)}\big|. \end{aligned}$$

The conclusive step is near:

$$\begin{aligned} \| C * C' \| &= \big|\sqrt{(a^2 + b^2) * (a'^2 + b'^2)}\big| \\ &= \big|\sqrt{(a^2 + b^2)}\big| * \big|\sqrt{(a'^2 + b'^2)}\big| \\ &= \| C \| * \| C' \| . \end{aligned}$$

and the theorem is finally proved for both of its parts: the absolute value and the norm of the complex numbers. □

Section: **Complex numbers** (2.5 Page 39 [6])

THEOREM 2.4. *Let $C = a + ci$, then*

$$\|C^{-1}\| = \|C\|^{-1}$$

and

$$Arg(C^{-1}) = -Arg(C).$$

[6] Ref. Sec. 2.5

Proof. Since we assumed that $C = a + ci \neq 0$, then by Eq. (2.5.21) in page 37

$$\begin{aligned} ||C^{-1}|| &= \left|\left|\left(\frac{a}{a^2+c^2}, -\frac{c}{a^2+c^2}\right)\right|\right| \\ &= \left|\sqrt{\left(\frac{a}{a^2+c^2}\right)^2 + \left(\frac{c}{a^2+c^2}\right)^2}\right|. \end{aligned}$$

Adding the fractions inside the square root symbol:

$$\begin{aligned} ||C^{-1}|| &= \left|\sqrt{\frac{(a^2+c^2)}{(a^2+c^2)^2}}\right| \\ &= \left|\sqrt{\frac{1}{(a^2+c^2)}}\right| \\ &= \left|\left|\frac{1}{a^2+c^2}\right|\right| \\ &= \left|\left|C\right|\right|^{-1}. \end{aligned}$$

Respect to the inverse of the argument of the complex C:

$$\begin{aligned} C^{-1} &= \tan^{-1}\left(\frac{\frac{-c}{a^2+c^2}}{\frac{a}{a^2+c^2}}\right) \\ &= \tan^{-1}\left(\frac{-c}{a}\right) \\ &= -\tan^{-1}\left(\frac{c}{a}\right) \\ &= -Arg(C). \end{aligned}$$

□

Section: **Complex numbers** (2.5 Page 40 [7])

THEOREM 2.5. *Let $C = a + ci$ and $C' = a' + c'i$ be any two complex numbers. Then:*

$$Arg(C * C') = Arg(C) + Arg(C')$$

and

$$Arg\left(\frac{C}{C'}\right) = Arg(C) - Arg(C').$$

[7] Ref. Sec. 2.5

Proof. We'll prove only the first part of the theorem. Is goes as follows:

$$\begin{aligned}
Arg(C * C') &= Arg\big((a+ci)*(a'+c'i)\big) \\
&= Arg\big(aa'-cc', (ac'+a'c)i\big) \\
&= \tan^{-1}\left(\frac{ac'+a'c}{aa'-cc'}\right).
\end{aligned}$$

On the other side:

$$\begin{aligned}
Arg(C) + Arg(C') &= Arg(a, ic) + Arg(a', ic') \\
&= \tan^{-1}\left(\frac{c}{a}\right) + \tan^{-1}\left(\frac{c'}{a'}\right).
\end{aligned}$$

By trigonometry:

$$\tan^{-1}\alpha_1 + \tan^{-1}\alpha_2 = \tan^{-1}\left(\frac{\alpha_1+\alpha_2}{1-\alpha_1*\alpha_2}\right).$$

Thus

$$\tan^{-1}\left(\frac{c}{a}\right) + \tan^{-1}\left(\frac{c'}{a'}\right) = \tan^{-1}\left(\frac{\frac{c}{a}+\frac{c'}{a'}}{1-\frac{cc'}{aa'}}\right).$$

By simple algebraic manipulations, it can be seen that:

$$\frac{ac'+a'c}{aa'-cc'} = \frac{\frac{c}{a}+\frac{c'}{a'}}{1-\frac{cc'}{aa'}}.$$

Hence:

$$Arg(C*C') = Arg(C) + Arg(C').$$

□

7.3 Theorems from Chapter 3: Transcomplex Numbers

Section: **Transcomplex numbers** (3.2 Page 44 [8])

THEOREM 3.1. *Two transcomplex numbers* $T = (a, b, c, d)$ *and* $T' = (a', b', c', d')$ *are equal if and only if* $a = a'$, $b = b'$, $c = c'$, *and* $d = d'$.

Proof. First we prove that if two transcomplex numbers are equal, then its elements are equal.

Since we are assuming that T and T' are transcomplex numbers, then T and T' are also ordered pairs. Therefore,

$$T = \Big\{ \Big\{ (a, b), \big\{ (a, b), (c, d) \big\} \Big\} \Big\}$$

and

$$T' = \Big\{ \Big\{ (a', b'), \big\{ (a', b'), (c', d') \big\} \Big\} \Big\}.$$

By the equality of ordered pairs,

$$(a, b) = (a', b') \qquad \text{and} \qquad (c, d) = (c', d').$$

Again, the equality of the ordered pairs (a, b) and (a', b') means that $a = a'$ and $b = b'$. By the same reasoning, the equality of the ordered pairs (c, d) and (c', d') means that $c = c'$ and $d = d'$. Hence, the first part of the theorem is proved.

Now we prove that if $a = a'$, $b = b'$, $c = c'$, and $d = d'$ then the two transcomplex numbers T and T' are equal.

Suppose that T and T' are not equal. Then at least one of the following inequalities hold: $a \neq a'$, $b \neq b'$, $c \neq c'$, or $d \neq d'$. But this is a contradiction since we assumed that $a = a'$, $b = b'$, $c = c'$, and $d = d'$, therefore, T and T' are equal. □

Section: **Transcomplex numbers** (3.2 Page 45 [9])

THEOREM 3.2. *Let* T *be any transcomplex number* $T = (a, b, c, d)$, *then*

$$T = a + b\,\tilde{} + ci + (di)\,\tilde{}.$$

[8] Ref. Sec. 3.2

[9] Ref. Sec. 3.2

Proof. By previous definitions:

$$T = (a, b) + (c, d)i.$$

But

$$(a, b) = (a, 0) + (0, b) = a + b\tilde{}$$

and

$$(c, d)i = (ci, 0) + (0, di) = ci + (di)\tilde{}.$$

Hence,

$$T = (a, b) + (c, d)i = a + b\tilde{} + ci + (di)\tilde{}.$$

□

Section: **Operations with transcomplex numbers** (3.3 Page 47 [10])

THEOREM 3.3. *Let* $T = (a, b, c, d)$ *and* $T' = (a', b', c', d')$. *Then*

$$\|T\| * \|T'\| = \|T * T'\|.$$

Proof.

$$\|T\| = \left(\left|\sqrt{a^2 + c^2}\right|, \left|\sqrt{b^2 + d^2}\right|\right)$$

and

$$\|T'\| = \left(\left|\sqrt{a'^2 + c'^2}\right|, \left|\sqrt{b'^2 + d'^2}\right|\right).$$

But

$$\begin{aligned}
\|T\| * \|T'\| &= \left(\left|\sqrt{a^2 + c^2}\right|, \left|\sqrt{b^2 + d^2}\right|\right) * \left(\left|\sqrt{a'^2 + c'^2}\right|, \left|\sqrt{b'^2 + d'^2}\right|\right) \\
&= \left(\left|\sqrt{a^2 + c^2}\right| * \left|\sqrt{a'^2 + c'^2}\right|, \left|\sqrt{b^2 + dc^2}\right| * \left|\sqrt{b'^2 + d'^2}\right|\right) \\
&= \left(\left|\sqrt{(a^2 + c^2) * (a'^2 + c'^2)}\right|, \left|\sqrt{(b^2 + d^2) * (b'^2 + d'^2)}\right|\right) \\
&= \left(\left|\sqrt{(aa')^2 + (ac')^2 + (a'c)^2 + (cc')}\right|, \left|\sqrt{(bb'')^2 + (bd')^2 + (b'd)^2 + (dd')}\right|\right).
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\|T * T'\| &= \left(\left|\sqrt{(aa' - cc')^2 + (ac' - a'c)^2}\right|, \left|\sqrt{(bb' - dd')^2 + (bd' - b'd)^2}\right|\right) \\
&= \left(\left|\sqrt{(aa')^2 + (cc')^2 + (ac')^2 + (a'c)^2}\right|, \left|\sqrt{(bb')^2 + (dd')^2 + (bd')^2 + (d'b)^2}\right|\right) \\
&= \left(\left|\sqrt{(aa')^2 + (ac')^2 + (a'c)^2 + (cc')^2}\right|, \left|\sqrt{(bb')^2 + (bd')^2 + (b'd)^2 + (dd')^2}\right|\right)
\end{aligned}$$

[10] Ref. Sec. 3.3

So that, clearly,

$$\|T\| * \|T'\| = \|T * T'\|.$$

□

Section: **Operations with transcomplex numbers** (3.3 Page 48 [11])

THEOREM 3.4. *Let $T = (a, b, c, d)$ and $T' = (a', b', c', d')$. Then*

$$Arg(T * T') = Arg(T) + Arg(T').$$

Proof.

$$\begin{aligned} Arg(T) &= \tan^{-1} \frac{c, d}{a, b} \\ &= \tan^{-1} \left(\frac{c}{a}, \frac{d}{b} \right) \\ &= \left(\tan^{-1} \frac{c}{a}, \tan^{-1} \frac{d}{b} \right) \end{aligned}$$

and

$$\begin{aligned} Arg(T') &= \tan^{-1} \frac{c', d'}{a', b'} \\ &= \tan^{-1} \left(\frac{c'}{a'}, \frac{d'}{b'} \right) \\ &= \left(\tan^{-1} \frac{c'}{a'}, \tan^{-1} \frac{d'}{b'} \right) \end{aligned}$$

so that

$$Arg(T) + Arg(T') = \left(\tan^{-1} \frac{c}{a} + \tan^{-1} \frac{c'}{a'}, \tan^{-1} \frac{d}{b} + \tan^{-1} \frac{d'}{b'} \right).$$

On the other hand,

$$\begin{aligned} Arg(T * T') &= \tan^{-1} \frac{ac' + a'c, bd' + b'd}{aa' - cc', bb' - dd'} \\ &= \left(\tan^{-1} \frac{ac' + a'c}{aa' - cc}, \tan^{-1} \frac{bd' + b'd}{bb' - dd'} \right). \end{aligned}$$

By a trigonometric equality:

$$\tan^{-1} \frac{c}{a} + \tan^{-1} \frac{c'}{a'} = \tan^{-1} \frac{ac' + a'c}{aa' - cc'}$$

and

$$\tan^{-1} \frac{d}{b} + \tan^{-1} \frac{d'}{b'} = \tan^{-1} \frac{bd' + b'd}{bb' - dd'}.$$

[11]Ref. Sec. 3.3

Therefore

$$\Big(\tan^{-1} ac' + a'caa' - cc', \tan^{-1} bd' + b'dbb' - dd'\Big) = \Big(\tan^{-1} \frac{c}{a} + \tan^{-1} \frac{c'}{a'}, \tan^{-1} \frac{d}{b} + \tan^{-1} \frac{d'}{b'}\Big).$$

Hence,

$$Arg(T * T') = Arg(T) + ARtg(T').$$

□

Section: **Operations with transcomplex numbers** (3.3 Page 48 [12])

THEOREM 3.5. *Let $T = (a, b, c, d)$ be a transcomplex number. Then*

$$T = \|T\| * \Big(\cos\big(Arg(T)\big), \sin\big(Arg(T)\big)\Big).$$

Proof. By a previous definition, we know that

$$\begin{aligned} \|T\| &= \Big(\sqrt{a^2 + c^2}, \sqrt{b^2 + d^2}\Big) \\ &= (r_1, r_2). \end{aligned}$$

From the geometric point of view, r_1 is the hypotenuse of a Pythagorean triangle with sides c and a. Analogously, r_2 is the hypotenuse of a Pythagorean triangle with sides d and b. Also, treating the norm of T as a full transcomplex number we write:

$$\|T\| = (r_1, r_2, 0, 0).$$

Respect to the argument:

$$\begin{aligned} Arg(T) &= \tan^{-1} \frac{(c, d)}{(a, b)} \\ &= \Big(\tan^{-1} \frac{c}{a}, \tan^{-1} \frac{d}{b}\Big) \\ &= (\theta_1, \theta_2) \end{aligned}$$

where

$$\theta_1 = \tan^{-1} \frac{c}{a}$$

and

$$\theta_2 = \tan^{-1} \frac{d}{b}.$$

[12]Ref. Sec. 3.3

Again, from the geometric point of view, θ_1 is the angle comprised in a Pythagorean triangle with sides c and a opposite to the angle θ_1. Analogously, θ_2 is the angle comprised in a Pythagorean triangle with sides d and b opposite to the angle θ_2. Therefore,

$$\begin{aligned}\cos Arg(T) &= \cos(\theta_1, \theta_2)\\ &= (\cos\theta_1, \cos\theta_2).\end{aligned}$$

But,

$$\cos\theta_1 = \frac{a}{\sqrt{a^2+c^2}}$$

and

$$\cos\theta_2 = \frac{b}{\sqrt{b^2+d^2}}.$$

Similarly,

$$\begin{aligned}\sin Arg(T) &= \sin(\theta_1, \theta_2)\\ &= (\sin\theta_1, \sin\theta_2).\end{aligned}$$

On the other hand,

$$\sin\theta_1 = \frac{c}{\sqrt{a^2+c^2}}$$

and

$$\sin\theta_2 = \frac{d}{\sqrt{b^2+d^2}}.$$

Therefore,

$$\cos Arg(T) = \left(\frac{a}{\sqrt{a^2+c^2}}, \frac{b}{\sqrt{a^2+c^2}}\right)$$

and

$$\sin Arg(T) = \left(\frac{c}{\sqrt{a^2+c^2}}, \frac{d}{\sqrt{a^2+c^2}}\right).$$

Hence,

$$\|T\| * \big(\cos Arg(T), \sin Arg(T)\big) = (r_1, r_2, 0, 0) * (\cos\theta_1, \cos\theta_2, \sin\theta_1, \sin\theta_2).$$

Realizing the transcomplex multiplication defined above, we obtain:

$$\begin{aligned}(r_1, r_2, 0, 0) * (\cos\theta_1, \cos\theta_2, \sin\theta_1, \sin\theta_2) = (&r_1\cos\theta_1 - 0 * \sin\theta_1,\\ &r_2\cos\theta_2 - 0 * \sin\theta_2,\\ &r_1\sin\theta_1 + 0 * \cos\theta_1,\\ &r_2\sin\theta_2 + 0 * \cos\theta_2).\end{aligned}$$

Using the trigonometric equalities of the preceding lines, substituting r_1, r_2, $\cos\theta_1$, $\cos\theta_2$, $\sin\theta_1$, and $\sin\theta_2$, we obtain:

$$\begin{aligned}(r_1\cos\theta_1, r_2\cos\theta_2, r_1\sin\theta_1, r_2\sin\theta_2) = \Big(&\sqrt{a^2+c^2} * \frac{a}{\sqrt{a^2+c^2}},\\ &\sqrt{b^2+d^2} * \frac{a}{\sqrt{b^2+d^2}},\\ &\sqrt{a^2+c^2} * \frac{c}{\sqrt{a^2+c^2}},\\ &\sqrt{b^2+d^2} * \frac{d}{\sqrt{b^2+d^2}}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\|T\| * \big(\cos Arg(T), \sin Arg(T)\big) &= (a,b,c,d)\\ &= T.\end{aligned}$$

□

Section: **The transnorm and transargument** (3.4 Page 50 [13])

THEOREM 3.6. *Let* $T = (a,b,c,d)$ *and* $T = (a',b',c',d')$. *Then,*

$$^{+}\|T\| \; * \; ^{+}\|T'\| \; \neq \; ^{+}\|T * T'\|.$$

Proof.

$$\begin{aligned}^{+}\|T\| &= \left\|\left(\left|\sqrt{a^2+c^2}\right|, \left|\sqrt{b^2+d^2}\right|\right)\right\|\\ &= \left|\sqrt{\left(|\sqrt{a^2+c^2}|\right)^2} + \sqrt{\left(|\sqrt{b^2+d^2}|\right)^2}\right|\\ &= \left|\sqrt{a^2+b^2+c^2+d^2}\right|.\end{aligned}$$

Similarly,

$$^{+}\|T'\| = \left|\sqrt{a'^2+b'^2+c'^2+d'^2}\right|.$$

Thus,

$$^{+}\|T\| \; * \; ^{+}\|T'\| \; = \left|\sqrt{\left(a^2+b^2+c^2+d^2\right) * \left(a^2+b^2+c^2+d^2\right)}\right|.$$

When the multiplication inside the radicand is performed, it will contain 16 terms.

[13] Ref. Sec. 3.4

Respect to the right part of the inequality that we are proving let

$$A = (aa')^2 + (cc')^2 + (ac')^2 + (a'c)^2$$

and

$$B = (bb')^2 + (dd')^2 + (bd')^2 + (b'd)^2.$$

Then,

$$^+\|T\| \;*\; ^+\|T'\| \;= \left|\sqrt{A+B}\right|.$$

This radicand contains an 8-term expression. Note that all 8 terms are contained in the radicand of $^+\|T\| \;*\; ^+\|T'\|$ which as mentioned contains 16 terms.

Therefore:

$$^+\|T\| \;*\; ^+\|T'\| \;\neq\; ^+\|T * T'\|.$$

□

Section: **The transnorm and transargument** (3.4 Page 50 [14])

THEOREM 3.7. *Let $T = (a, b, c, d)$ and $T = (a', b', c', d')$. Then,*

$$^+Arg(T * T') \neq\; ^+Arg(T) +^+ Arg(T'). \tag{3.3.1}$$

Proof.

$$\begin{aligned} ^+Arg(T) &= \|Arg(T)\| \\ &= \|(\theta_1, \theta_2)\| \\ &= \left|\sqrt{(\theta_1)^2 + (\theta_2)^2}\right|. \end{aligned}$$

In the same way,

$$^+Arg(T') = \left|\sqrt{(\theta_1')^2 + (\theta_2')^2}\right|.$$

And

$$\begin{aligned} ^+Arg(T * T') &= ||\big(\psi_1, \psi_2\big)|| \\ &= \left|\sqrt{(\psi_1)^2 + (\psi_2)^2}\right| \end{aligned}$$

where

$$\psi_1 = \theta_1 + \theta_1' \qquad \text{and} \qquad \psi_2 = \theta_2 + \theta_2'.$$

[14] Ref. Sec. 3.4

Thus

$$\left|\sqrt{(\psi_1)^2 + (\psi_2)^2}\right| = \left|\sqrt{(\theta_1 + \theta_1')^2 + (\theta_2 + \theta_2')^2}\right|.$$

Also:

$$^+Arg(T) +^+ Arg(T') = \left|\sqrt{(\theta_1)^2 + (\theta_2)^2}\right| + \left|\sqrt{(\theta_1')^2 + (\theta_2')^2}\right|.$$

It is evident that:

$$\left|\sqrt{(\theta_1)^2 + (\theta_2)^2}\right| + \left|\sqrt{(\theta_1')^2 + (\theta_2')^2}\right| \neq \left|\sqrt{(\theta_1)^2 + (\theta_2)^2 + (\theta_1')^2 + (\theta_2')^2}\right|.$$

Therefore,

$$^+Arg(T * T') \neq \ ^+Arg(T) +^+ Arg(T'). \tag{3.3.2}$$

□

Section: **Exponentiation of transcomplex numbers** (3.5 Page 50 [15])

THEOREM 3.8. *Let $T = (a, b, c, d)$ be a transcomplex number. For short, let*

$$R =^+ \left\|\|T\|\right\| = (r_1, r_2)$$

and

$$W =^+ Arg(T) = \left\|Arg(T)\right\| = (\Theta_1, \Theta_2)$$

then

$$T = Re^{iW}.$$

where e is the natural logarithm base and i the imaginary unit.

Proof. Recall that

$$r_1 = \left|\sqrt{a^2 + c^2}\right| \qquad \text{and} \qquad r_2 = \left|\sqrt{b^2 + d^2}\right|$$

and

$$\theta_1 = \tan^{-1}\frac{c}{a} \qquad \text{and} \qquad \theta_2 = \tan^{-1}\frac{d}{b}.$$

That is, a, c, and r_1 make a rectangular triangle with hypotenuse r_1. The same happens for b, d, and r_2. Then

$$a = r_1 \cos\theta_1, \qquad b = r_2 \cos\theta_2$$

$$c = r_1 \sin\theta_1, \qquad d = r_2 \sin\theta_2.$$

[15] Ref. Sec. 3.5

Therefore

$$\begin{aligned}T &= (a, b, c, d)\\ &= (r_1 \cos\theta_1, b = r_2 \cos\theta_2, c = r_1 \sin\theta_1, r_2 \sin\theta_2).\end{aligned}$$

Then

$$T = (r_1 \cos\theta_1, r_2 \cos\theta_2 + i(r_1 \sin\theta_1, r_2 \sin\theta_2).$$

By a previous definition, the above equation can be rewritten as:

$$T = (r_1, r_2) * (\cos\theta_1, \cos\theta_2) + i\big((r_1, r_2) * (\sin\theta_1, \sin\theta_2)\big).$$

Taking out the common factor (r_1, r_2) we have:

$$T = (r_1, r_2) * \big((\cos\theta_1, \cos\theta_2), (\sin\theta_1, \sin\theta_2)\big).$$

But, because

$$\cos(\theta_1, \theta_1) = (\cos\theta_1, \cos\theta_2)$$

and

$$\sin(\theta_1, \theta_1) = (\sin\theta_1, \sin\theta_2)$$

then

$$T = (r_1, r_2) * \big(\cos(\theta_1, \theta_2), \sin(\theta_1, \theta_2)\big).$$

To continue with the proof, now we use a well known trigonometric formula that involves the imaginary unit i and the natural constant e, not proved in this theory, and a previous result of previous sections. The trigonometric formula is:

$$e^{i\Omega} = \cos\Omega + i \sin\Omega.$$

The previous result is that if for a function f with real domain, $f(a)$ and $f(b)$ exists, then $f(a, b)$ also exist, and is $(f(a), f(b)$.

So,

$$e^{i(\theta_1, \theta_2)} = \cos(\theta_1, \theta_2) + i \sin(\theta_1, \theta_2).$$

Hence,

$$T = (r_1, r_2)e^{i(\theta_1, \theta_2)}.$$

But, since $(r_1, r_2) = R$ and $(\theta_1, \theta_2) = W$ we now finally have:

$$T = Re^{iW}.$$

The same result, but using the full transnorm and transargument notation can be rewritten as:

$$T =^{+} \left|\left|T\right|\right| e^{i\left({}^{+}Arg(T)\right)}.$$

□

Section: **The transcomplex numbers field** (3.7 Page 52 [16])

THEOREM 3.9. *The transcomplex number class $\mathbb{T}$, together with the addition and multiplication operations defined for this class make a field.*

Proof. Let $T = (a, b, c, d)$, $T' = (a', b', c', d')$, and $T'' = (a'', b'', c'', d'')$ be any three transcomplex numbers.

1. $+$ is closed in $\mathbb{T}$ because

$$\begin{aligned} T + T' &= (a, b, c, d) + (a', b', c', d') \\ &= (a + a', b + b', c + c', d + d'). \end{aligned}$$

and $a + a'$, $b + b'$, $c + c'$ and $d + d'$ always exist since they are real numbers.

2. $+$ is associative on $\mathbb{T}$ because

$$\begin{aligned} (T + T') + T'' &= \big((a, b, c, d) + (a', b', c'd')\big) + (a'', b'', c'', d'') \\ &= (a + a', b + b', c + c', d + d') + (a'', b'', c'', d'') \\ &= (a + a' + a'', b + b' + b'', c + c' + c'', d + d' + d'') \\ &= (a, b, c, d) + \big((a', b', c', d') + (a'', b'', c'', d'')\big) \\ &= T + (T' + T''). \end{aligned}$$

3. $+$ is commutative on $\mathbb{T}$ because

$$\begin{aligned} T + T' &= (a, b, c, d) + (a', b', c', d') \\ &= (a + a', b + b', c + c', d + d') \\ &= (a' + a, b' + b, c' + c, d' + d) \\ &= T' + T. \end{aligned}$$

4. The identity unit element of addition on $\mathbb{T}$ is the transcomplex number

$$T' = (0, 0, 0, 0)$$

[16] Ref. Sec. 3.7

because

$$\begin{aligned} T + T' &= (a, b, c, d) + (0, 0, 0, 0) \\ &= (a + 0, b + 0, c + 0, d + 0) \\ &= (a, b, c, d) \\ &= T. \end{aligned}$$

5. The negative of every transcomplex number $T = (a, b, c, d)$ of $\mathbb{T}$ is the transcomplex number $-T = (-a, -b, -c, -d)$ because

$$\begin{aligned} T + (-T) &= (a, b, c, d) + (-a, -b, -c, -d) \\ &= (a - a, b - b, c - c, d - d) \\ &= (0, 0, 0, 0). \end{aligned}$$

6. The operation of multiplication * is closed on $\mathbb{T}$ because

$$T * T' = (aa' - cc', bb' - dd', ac' + ca', bd' + db')$$

and the real numbers $aa' - cc'$, $bb' - dd'$, $ac' + ca'$, and $bd' + db'$ are guaranteed to exist.

7. The operation of multiplication * is associative on $\mathbb{T}$, that is:

$$(T * T') * T'' = T * (T' * T'').$$

The proof of this theorem is a bit cumbersome, so to avoid repetitive algebraic manipulations, we'll only prove that the first entry of the ordered pair representation of $(T * T') * T''$ is equal to the first entry of the ordered pair representation of $T * (T' * T'')$.

Let's first compute $(T * T') * T''$

$$\begin{aligned} (T * T') * T'' &= (aa' - cc', bb' - dd', ac' + ca', bd' + db') * T'' \\ &= (A, B, C, D) * (a'', b'', c'', d'') \\ &= (Aa'' - Cc'', Bb'' - Dd'', Ac'' + Ca'', Bd'' + Db''). \end{aligned}$$

Where

$$(A, B, C, D) = (aa' - cc', bb' - dd', ac' + ca', bd' + db').$$

Note that the first entry of $(T * T') * T''$ is $Aa' - Cc'$.

On the other side

$$\begin{aligned} T * (T' * T'') &= T * (a'a'' - c'c'', b'b'' - d'd'', a'c'' + c'a'', b'd'' + d'b'') \\ &= (a, b, c, d) * (A', B', C', D') \\ &= aA' - cC', bB' - dD', Ac' + Ca', Bd' + Db'). \end{aligned}$$

Note that here the first entry of $T * (T' * T'')$ is $aA' - cC'$.

Now we have to show that

$$Aa' - Cc' = aA' - cC'$$

but this is only simple algebraic manipulation because

$$A = aa' - cc'$$
$$A' = a'a'' - c'c''$$
$$C = ac' + ca'$$
$$C' = a'c'' + c'a''.$$

Hence,

$$\begin{aligned} Aa'' - Cc'' &= (aa' - cc')a'' - (ac' + ca')c'' \\ &= aa'a'' - a''cc' - ac'c'' - a'cc''. \end{aligned}$$

$$\begin{aligned} aA' - cC' &= a(a'a'' - c'c'') - c(a'c'' + c'a'') \\ &= aa'a'' - ac'c'' - a'cc'' - cc'a'' \\ &= aa'a'' - cc'a'' - ac'c'' - a'cc'' \\ &= Aa'' - Cc''. \end{aligned}$$

Therefore, the first entry of the ordered pair representation of $(T * T') * T''$ is equal to the first entry of the ordered pair representation of $T * (T' * T'')$ as was our purpose to demonstrate.

However, the full demonstration that $(T * T') * T'' = T * (T' * T'')$ requires to go four times trough similar algebraic manipulations, but all of them will show that the n-th entry of $(T * T') * T''$ is equal to the n-th entry of $T * (T' * T'')$, for n= 1, 2, 3, 4.

8. The operation of multiplication * is commutative on $\mathbb{T}$.

This is so because

$$T * T' = (aa' - cc', bb' - dd', ac' + ca', bd' + db')$$

and

$$T' * T = (a'a - c'c, b'b - d'd, a'c + c'a, b'd + d'b).$$

However, each one of the elements of $T * T'$ is respectively equal to each one of the elements of $T' * T$, because $aa' - cc' = a'a - c'c$, etc. That is, each n-th entry of $T * T'$ is equal to the n-th entry of $T' * T$.

9. The identity element under multiplication of transcomplex numbers is the transcomplex number

$$T' = (1, 1, 0, 0)$$

because

$$\begin{aligned} T * (1, 1, 0, 0) &= (a * 1, -c * 0, b * 1 - d * 0, a * 0 + c * 1, b * 0 + d * 1) \\ &= (a - 0, b - 0, 0 + c, 0 + d) \\ &= (a, b, c, d) \\ &= T. \end{aligned}$$

10. For each transcomplex number $T = (a, b, c, d)$ there is a T^{-1} called the inverse of T such that $T * T^{-1} = (1, 1, 0, 0)$.

Let T' be the transcomplex number that is the inverse of T, that is, let $T' = T^{-1}$. Then

$$\begin{aligned} T * T^{-1} &= (aa' - cc', bb' - dd', ac' + ca', bd' + db') \\ &= (1, 1, 0, 0). \end{aligned}$$

Hence,

$$\begin{aligned} aa' - cc' &= 1 \\ bb' - dd' &= 1 \\ ac' + ca' &= 0 \\ bd' + db' &= 0. \end{aligned}$$

Now we need to find the four values of the entries of T^{-1}. To find a' we solve for ac' in the equation $aa' - cc' = 1$, and this gives:

$$\begin{aligned} ac' - ca' &= 1 \\ -cc' &= 1 - aa' \\ c' &= \frac{1 - aa'}{-c} \\ ac' &= a\Big(\frac{1 - aa'}{-c}\Big) \\ &= \frac{a - a^2}{a'} - c. \end{aligned}$$

Now we substitute the ac' value in the equation $ac' + ca' = 0$ to obtain

$$\begin{aligned} ac' + ca' &= 0 \\ \frac{a - a^2 a'}{-c} &= 0. \end{aligned}$$

Multiplying by $-c$ both sides of the equation we obtain:

$$\begin{aligned} a - a^2a' - c^2a' &= 0 \\ -a^2a' - c^2a' &= -a \\ a^2a' + c^2a' &= a \\ a'\left(a^2 + c^2\right) &= a. \end{aligned}$$

Finally, we have the value of a' which is the first entry of T^{-1}.

$$a' = \frac{a}{a^2 + c^2}$$

Now we use the equation $ac' + ca'$ to find the value of c' as follows:

$$\begin{aligned} ac' + ca' &= 0 \\ ac' &= -ca'. \end{aligned}$$

Therefore

$$\begin{aligned} c' &= \frac{-c}{a}a' \\ &= \frac{-c}{a}\left(\frac{a}{a^2 + c^2}\right) \\ &= \frac{-c}{a^2 + c^2}. \end{aligned}$$

Two values are remaining: b' and d'. To find b', we solve for bd' in the equation $bb' - dd' = 1$ and this gives:

$$\begin{aligned} bb' - dd' &= 1 \\ -dd' &= 1 - bb' \\ d' &= \frac{1 - bb'}{-d}. \end{aligned}$$

Multiplying by b both sides of the last equality, we have

$$\begin{aligned} bd' &= b\left(\frac{1 - bb'}{-d}\right) \\ &= \frac{b - b^2b'}{-d}. \end{aligned}$$

Now we substitute the bd' in the $bd' + db' = 0$ equation.

$$\begin{aligned} bd' + db' &= 0 \\ \frac{b - b^2b'}{-d} + db' &= 0. \end{aligned}$$

Multiplying by $-d$ both sides of this last equality, it is obtained:

$$\begin{aligned} b - b^2b' - d^2b' &= 0 \\ -b^2b' - d^2b' &= 0. \end{aligned}$$

Now we multiply by -1 both sides and factor by b'.

$$\begin{aligned} b^2b' + d^2b' &= b \\ b'\left(b^2 + d^2\right) &= b. \end{aligned}$$

Finally:

$$b' = \frac{b}{b^2 + d^2}.$$

Now we use the equation $bd' + db' = 0$ to find the value of d' as follows:

$$\begin{aligned} bd' + db' &= 0 \\ bd' &= -db'. \end{aligned}$$

Hence,

$$d' = \frac{-d}{b}(b').$$

Substituting for the previously found value b', we have:

$$\begin{aligned} d' &= -db(b') \\ &= \frac{-d}{b}\left(\frac{b}{b^2 + d}\right). \end{aligned}$$

Therefore, the inverse of any transcomplex number $T = (a, b, c, d) \neq 0$ is the transcomplex number T^{-1} given by:

$$T^{-1} = \left(\frac{a}{a^2 + c^2}, \frac{b}{b^2 + d^2}, \frac{-c}{a^2 + c^2}, \frac{-d}{a^2 + c^2}\right).$$

11. Multiplication of transcomplex numbers is distributive respect to addition.

We are going to do here the same we did in entry number 7 of this proof: to avoid repetitive and cumbersome algebraic manipulations, we'll only prove that the first entry of $T * (T' + T'')$ is equal to the first entry of $T * T' + T * T''$. To prove that the other entries are respectively equal, the procedure is the same as the one that follows.

In order that the multiplication be distributive, the following must be accomplished:

$$T * (T' + T'') = T * T' + T * T''.$$

But,

$$\begin{aligned} T * (T' + T'') &= T\big((a', b', c'd') + (a'', b'', c'', d'')\big) \\ &= (a, b, c, d) * (A, B.C, D) \\ &= (aA - cC, bB - dD, aC + cA, bD + dB) \end{aligned}$$

where

$$\begin{aligned} A &= a' + a'' \\ B &= b' + b'' \\ C &= c' + c'' \\ D &= d' + d''. \end{aligned}$$

Therefore, for the first entry we have:

$$\begin{aligned} aA - cC &= a(a' + a'') - c(c' + c'') \\ &= aa' + a'' - c' - c''. \end{aligned}$$

On the other side:

$$\begin{aligned} T * T' + T * T'' &= (aa' - cc', bb' - dd', ac' + ca', db' + bd') \\ &\qquad + (aa'' - cc'', bb'' - dd'', ac'' + ca'', db'' + bd'') \\ &= (aa' - cc' + aa'' - cc'', bb' - dd' + bb' - dd'', \\ &\qquad ac' + ca' + ac'' + ca'', db' + bd' + db'' + bd'). \end{aligned}$$

Now it is clearly seen that the first entry of $T * T' + T * T''$, which is $aa' - cc' + aa'' - cc''$ is equal to the $aA - cC$ above which also is $aa' + a'' - cc' - cc''$, when the terms are rearranged.

12. If $T * T' = 0$, then at least one of the following conditions holds:

$$T = 0, \qquad \text{or} \qquad T' = 0.$$

Let $T = (a, b, c, d)$ and $T' = (a', b', c', d')$ then

$$T * T' = (aa' - cc', bb' - dd', ac' + ca', bd' + db').$$

But, if $T * T' = 0$ then $T * T' = (0, 0, 0, 0)$.

It is clear that if $T = 0$ or $T' = 0$ then $T * T' = 0$, because $(aa' - cc', bb' - dd', ac' + ca', bd' + db')$ becomes $(0, 0, 0, 0)$ so we'll be looking for the converse case where if $T * T' = 0$ then $T = 0$ or $T' = 0$. In order that this case becomes true, the following must be true:

$$aa' - cc' = 0$$
$$bb' - dd' = 0$$
$$ac' + ca' = 0$$
$$bd' + db' = 0$$

but the above conditions are satisfied if

$$aa' = cc'$$
$$bb' = dd'$$
$$ac' = -ca'$$
$$bd' = -db'.$$

From the above equations now we take $aa' = cc'$ and $ac' = -ca'$, and solve for a' in both. Then we have:

$$a' = \frac{cc'}{a} \qquad \text{and} \qquad a' = \frac{a'c'}{c}.$$

Equating both, we obtain:

$$\frac{cc'}{a} = -\frac{ac'}{c}$$

which is equivalent to say that

$$\frac{c}{a} = -\frac{a}{c}$$

or

$$c^2 - a^2 = 0.$$

Obviously, this equation is satisfied if $c = a$, but is not the only one. If $c = a$, then we can substitute a for c in the $aa' = cc'$ equation to obtain $aa' = ac'$, which implies that $a' = c'$.

Similarly, it can be proved that $d = b$ and that $b' = d'$. Thus, in order that $T * T' = 0$ it must be true that if $T = (a, b, c, d)$ then $T = (a, b, a, b)$ and $T' = (a', b', c', d') = (a', b', a', b')$.

In summary, all this implies that

$$(a, b, a, b) * (a', b', a', b') = (aa' - aa', bb' - bb', aa' + aa', bb' + bb').$$

That $aa' - aa' = 0$ is true for any value of a and a'. Similarly, $bb' - bb' = 0$ for any value of b and b'. But $aa' + aa' = 0$ only if $a = 0$ and $a' = 0$, and $bb' + bb' = 0$ only for $b = 0$ and $b' = 0$. Thus, it is proved that $T * T' = 0$ if and only if $T = 0$ or $T' = 0$. $\square$

7.4 Theorems from Chapter 4: The Coordinate System S^4

Section: **The Space** S^4 (4.2 Page 58 [17])

THEOREM 4.1. *Respect to the orthogonality of the axes of the space S^4 the following two propositions are true:*

- *The four axes of the tetraspace S^4 are mutually orthogonal.*
- *No axis is orthogonal to itself.*

Proof. We first prove the first assertinon of the theorem. Using the definition of perpendicularity of transcomplex numbers the following axial units are perpendicular

$$1 * 1^{\~} = 0, \qquad 1 * i^{\~} = 0, \qquad 1^{\~} * i = 0, \qquad \text{and} i * i^{\~} = 0$$

because their real part is obviously zero.

On the other hand,

$$\Re(1 * i) = \Re(0, 0, i, 0) = (0, 0) = 0$$

and

$$\Re(1^{\~} * i^{\~}) = \Re(0, 0, 0, 1) = (0, 0) = 0.$$

Therefore

$$1 \perp 1^{\~}, 1 \perp i^{\~}, 1 \perp i,$$
$$1^{\~} \perp i, 1^{\~} \perp i^{\~}$$
$$i \perp i^{\~}.$$

Note that this covers all possible multiplications of product of axial units.

Now all we need is to prove is that if $1 \perp 1^{\~}$, that is, if the axial unit 1 is perpendicular to the axial unit $1^{\~}$, then any number a belonging to X-coordinate is also perpendicular to every number b belonging to the $Y^{\~}$-coordinate.

But if a number a belongs to the X-coordinate, then it is of the form

$$T = (a, 0, 0, 0) = a(1, 0, 0, 0)$$

[17] Ref. Sec. 4.2

and if b belongs to the $Y^{\sim}$-coordinate then it is of the form

$$T' = (0, b, 0, 0) = b(0, 1, 0, 0).$$

Therefore,

$$\begin{aligned} T * T' &= a(1, 0, 0, 0) * b(0, 1, 0, 0) \\ &= a1 * b1^{\sim} \\ &= ab(1 * 1^{\sim}). \end{aligned}$$

But $(1*1^{\sim}) = 0$, hence, $T \perp T'$ for any number T of X and any number T' of $Y^{\sim}$, therefore X is orthogonal to $Y^{\sim}$. Or, in symbols:

$$X \perp Y^{\sim}.$$

The same routine applies to demonstrate that all other axes are mutually orthogonal. At the end we should have:

$$X \perp Y^{\sim}, \quad X \perp i^{\sim} Z, \quad X \perp iZ$$
$$Y^{\sim} \perp iZ, \quad Y^{\sim} \perp i^{\sim} Z, \quad iZ \perp i^{\sim} Z.$$

The part B of the theorem states that no axis is orthogonal to itself.

That is clearly seen from the fact that

$$\begin{aligned} \Re(1 * 1) &= 1 \neq 0 \\ \Re(1^{\sim} * 1^{\sim}) &= 1^{\sim} \neq 0 \\ \Re(i * i) &= -1 \neq 0 \\ \Re(i^{\sim} * i^{\sim}) &= -1^{\sim} \neq 0. \end{aligned}$$

Thus, no axial unit is perpendicular to itself. With some little algebraic manipulations, this result can be extended to prove that no axis is orthogonal to itself. □

Section: **The Space** S^4 (4.2 Page 58 [18])

THEOREM 4.2. *No element of one axis can be plotted into another axis.*

Proof. Let $T = (a, b, c, d)$ be a point on the X-axis, and let $T' = (a', b', c', d')$ be a point on an unknown axis, say U, where U may be the $Y^{\sim}$-axis, the iZ-axis, or the $i^{\sim}Z$-axis.

If T belongs to the X-axis, then $T = (a, 0, 0, 0)$. If T belongs to U then $T = (0, b, 0, 0)$, or $T = (0, 0, c, 0)$, or $T = (0, 0, 0, d)$, but that is a contradiction, since we assumed that $T = (a, 0, 0, 0)$, hence, T cannot be plotted on any other axis. Thus $U = X$. □

[18] Ref. Sec. 4.2

7.5 Theorems from Chapter 5: Transcomplex Functions

Section: **Transcomplex functions** (5.3 Page 84 [19])

THEOREM 5.1. *Let F *be a transcomplex map from a domain A of complex variables $U = (x, 0, z)$ to a range B of transcomplex numbers $W = (x', y', z')$. Then*

$$F^*(x, z) = (x, f(x, z), g(x, z)).$$

Proof. Recall the definition of transcomplex map

$$F^*(U) = \Big(\Re(U), \Re\big(F(U)\big), \Re\big(\Im(U)\big)\Big)$$

but

$$\Re(U) = \Re(x, o, z, 0) = (x, 0) = x$$

and

$$F(x + iz) = f(x, z) + ig(x, z).$$

At the same time,

$$\Re\big(F(U)\big) = f(x, z) \qquad \text{and} \qquad \Im\big(F(U)\big) = ig(x, z)$$

hence

$$\Re\big(\Im(U)\big) = g(x, z).$$

Making the corresponding substitutions in the theorem premises we obtain

$$F^*(U) = \big(x, f(x, z), g(x, z)\big)$$

but since $U = x + iz$, it is finally obtained:

$$F^*(U) = \Big(\Re(U), \Re\big(F(U)\big), \Re\big(\Im(U)\big)\Big).$$

□

Section: **Transcomplex functions** (5.3 Page 85 [20])

[19] Ref. Sec. 5.3
[20] Ref. Sec. 5.3

THEOREM 5.2. *A transcomplex map $F^*(U)$ is a semicomplex map $F^{\sim}(U)$ plotted in the plane $X = x$. In symbols,*

$$F^*(U) = x + F^{\sim}(U).$$

Proof. It was already proved that

$$F^*(U) = \big(x, f(x,z), g(x,z)\big).$$

On the other hand,

$$\begin{aligned} F^{\sim}(U) &= \big(f(x,z)^{\sim} + ig(x,z)\big) \\ &= \big(0, f(x,z), g(x,z)\big). \end{aligned}$$

Therefore,

$$F^*(U) + x = \big(x, f(x,z), g(x,z)\big).$$

□

Section: **Transcomplex functions** (5.3 Page 86 [21])

THEOREM 5.3. *If the domain D of a transcomplex function is of real numbers only, then that transcomplex function reduces to its simple real function expression. That is:*

$$F^*(U) = f(x)$$

when $x \in \mathbb{R}$.

Proof. We already know that

$$F^*(x.z) = \big(x, f(x,z), g(x,z)\big).$$

But if the domain is of real numbers only, the $z = 0$ always, and the transformation reduces to

$$F^*(x,0) = \big(x, f(x,0), g(x,0)\big)$$

which is the same as

$$F^*(x) = \big(x, f(x), g(x)\big).$$

□

[21] Ref. Sec. 5.3

Section: **Transcomplex surfaces** (5.4 Page 88 [22])

THEOREM 5.4. *Let $U = (x, z)$ be a complex region on the complex plane and let $F(U) = f(x, z) + g(x, z)i$ be the complex function defined on the region U. Then the interception of the transcomplex map $F^*(U)$ with the $XY\tilde{}$plane is the same as the image-real map $f(x)$ defined for the subdomain of U of all ordered pair such that $z = 0$.*

Proof. When we talk of the subdomain of U of all ordered pair such that $z = 0$, we are strictly talking about the point on the X-axis that also belongs to U. This subdomain is also the one obtained by the intersection of the normal plane with the domain U. Another way of referring to that subdomain is to say that is the subset of U of all real numbers. Let us denote by U_R that subset.

What the theorem states then is that the transcomplex map $F^*(U)$ when applied to the subdomain U_R produces the real map $f(x)$.

For that subdomain, U_R, it is clear that $z = 0$ and then $U = (x, z) = F(x)$. Hence, the functions $f(x, z)$ and $g(x, z)$ are reduced to:

$$F(x) = f(x) + g(x)i.$$

The presence of an imaginary component on the preceding equation implies that even when the subdomain is of real numbers only, the mate of a real number can still be a full complex number. But what the theorem states is not that a real numbers subdomain produce necessarily real mates; what the theorem states is that the interception of a transcomplex map with the $XY\tilde{}$-plane is the product of real numbers only.

However, the $g(x)i$ component is the equivalent of plotting along the iZ-axis, $g(x)$ being the Z-component, but since we are only talking about the $XY\tilde{}$-plane, that Z-component is not taken in account for our purposes. Hence, what is left is that $F(x) = f(x)$. □

[22] Ref. Sec. 5.4

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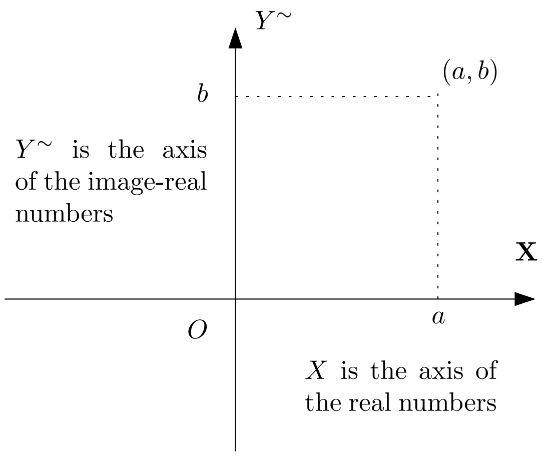

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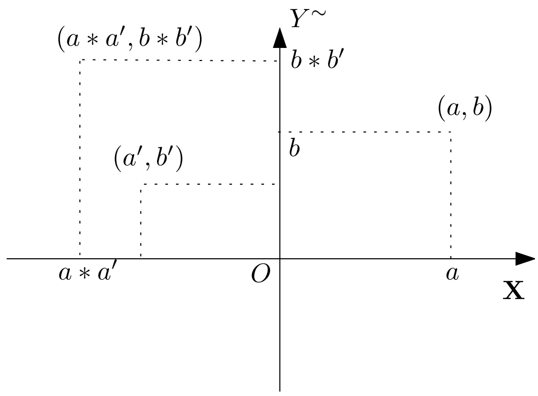

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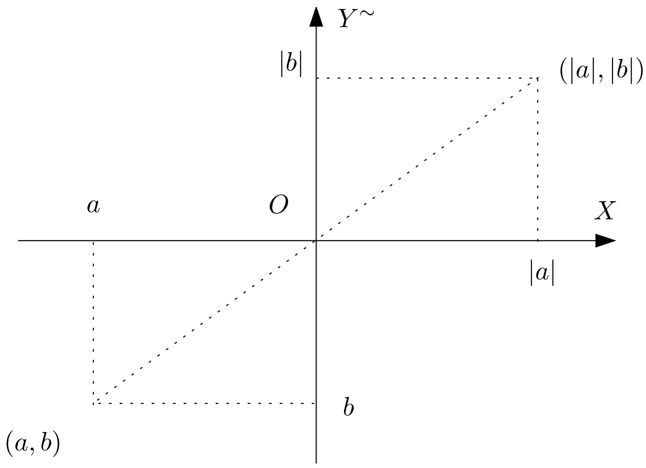

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[24] Ref. Sec. 1.4
[25] Ref. Sec. 1.5

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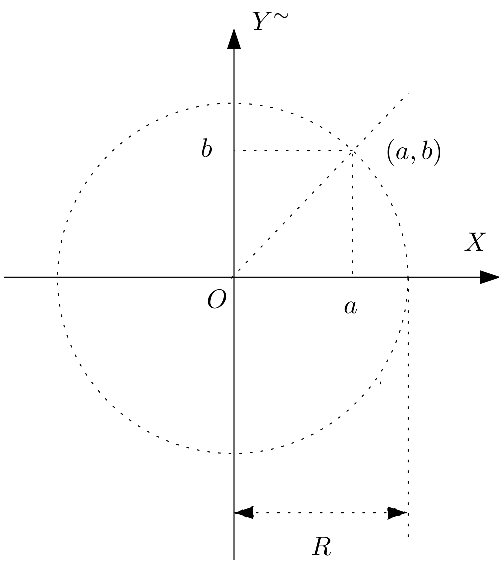

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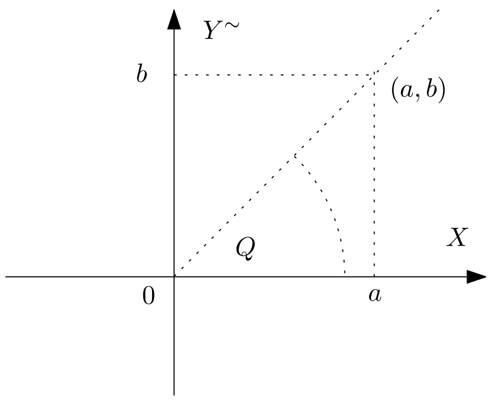

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[26] Ref. Sec. 1.5

[27] Ref. Sec. 1.5

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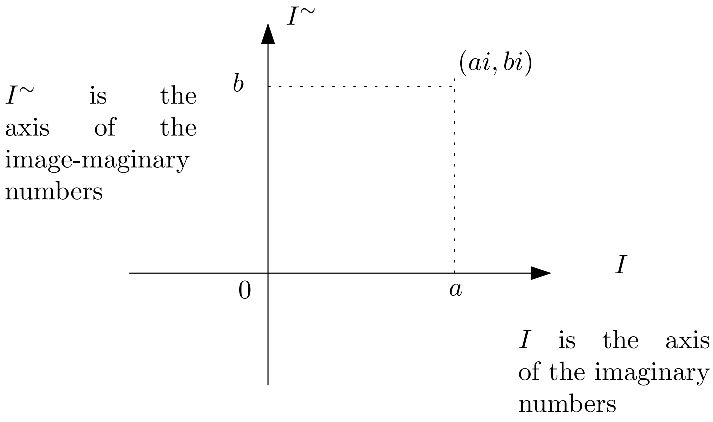

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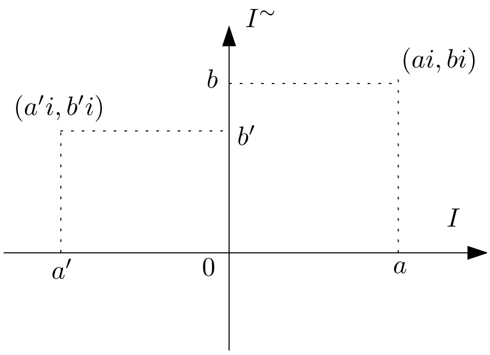

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[28]Ref. Sec. 2.3
[29]Ref. Sec. 2.3

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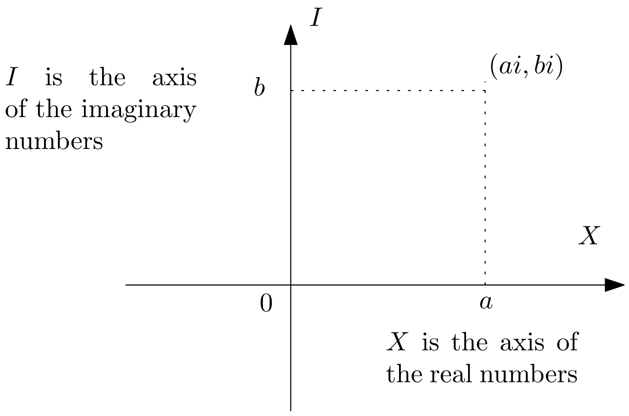

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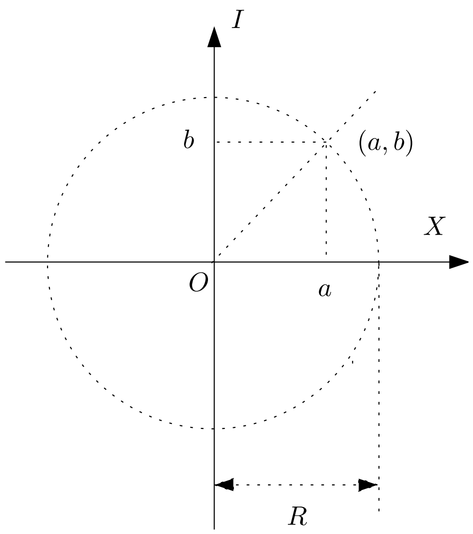

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[30]Ref. Sec. 2.5
[31]Ref. Sec. 2.5

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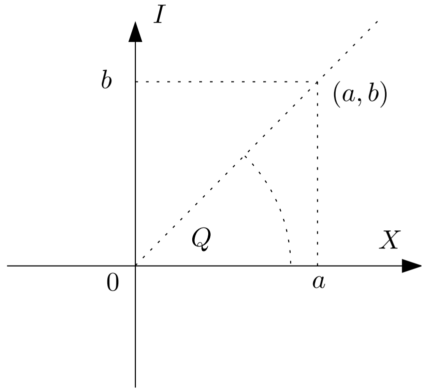

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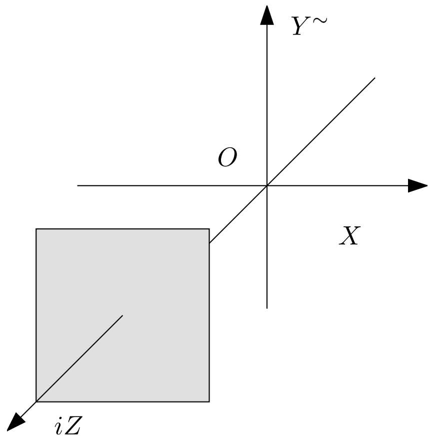

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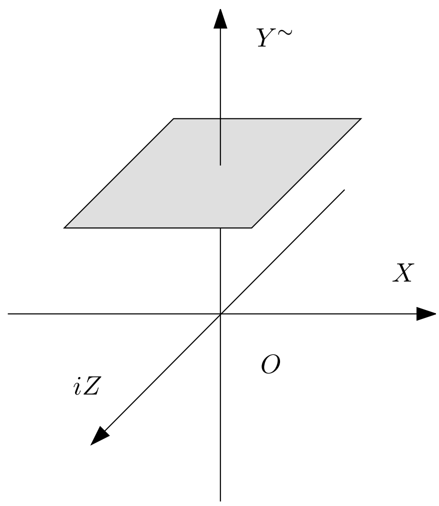

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[33] Ref. Sec. 4.3
[34] Ref. Sec. 4.3

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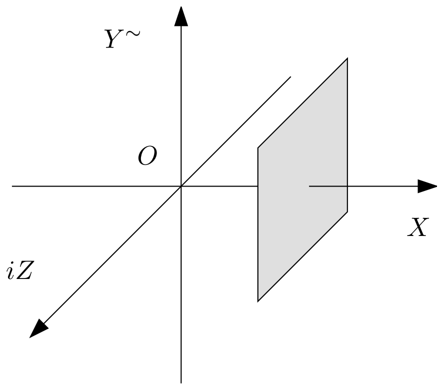

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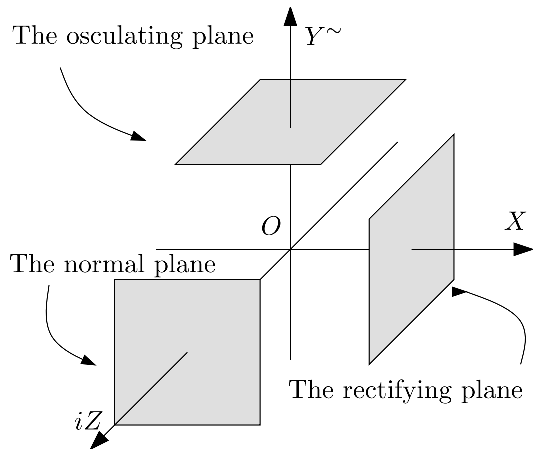

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[35] Ref. Sec. 4.3
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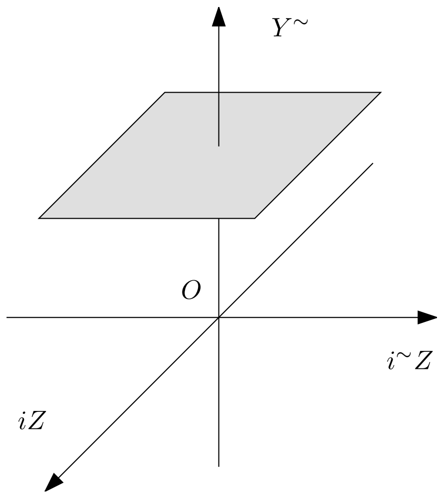

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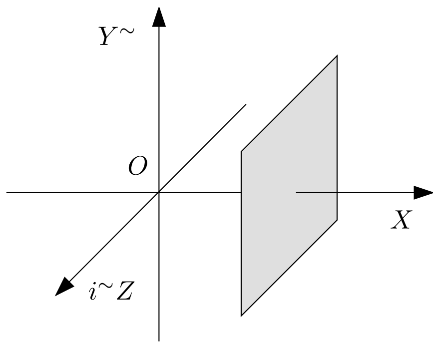

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[37] Ref. Sec. 4.3
[38] Ref. Sec. 4.3

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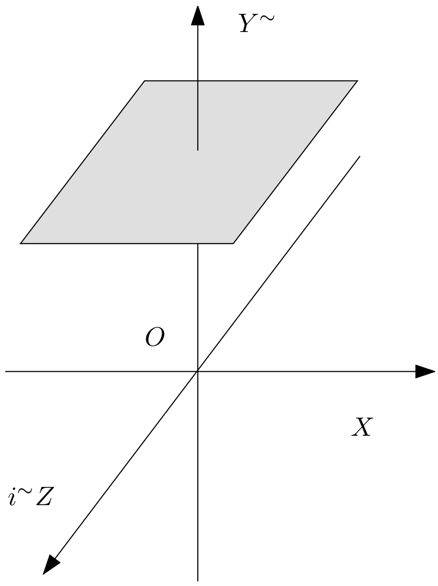

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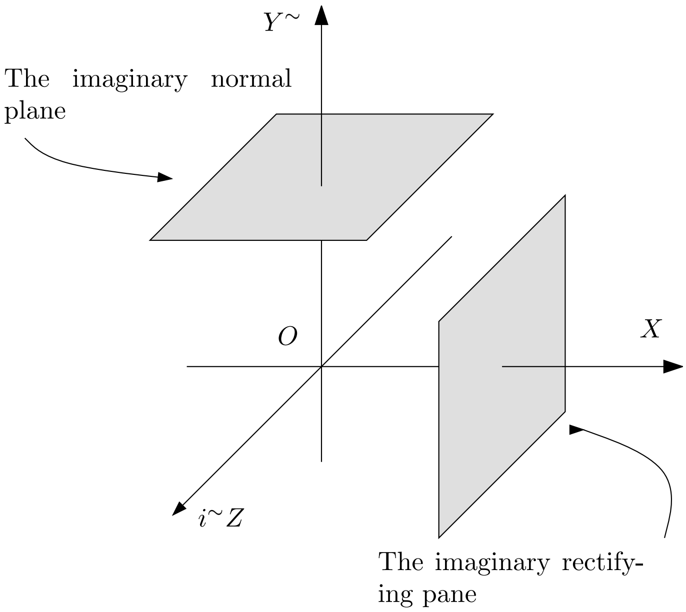
[39]Ref. Sec. 4.3
[40]Ref. Sec. 4.3



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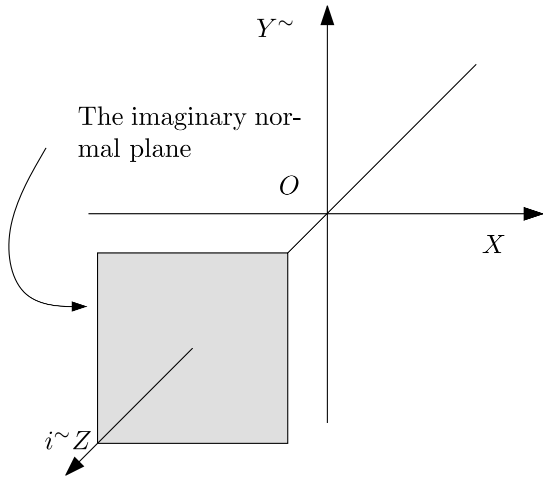

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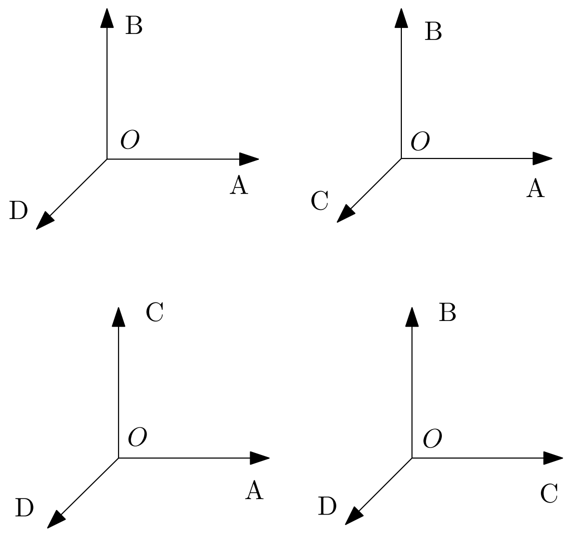
[41]Ref. Sec. 4.3
[42]Ref. Sec. 4.3



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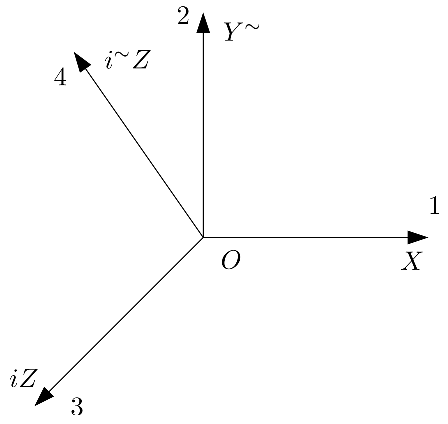

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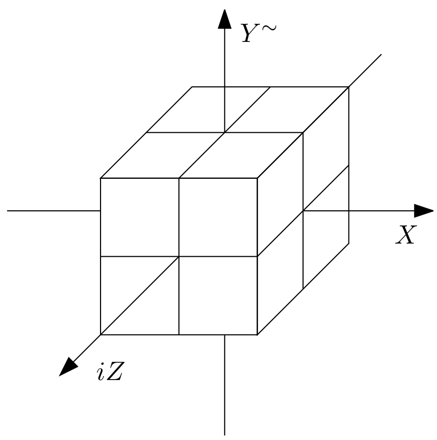

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[43] Ref. Sec. 4.3
[44] Ref. Sec. 4.3

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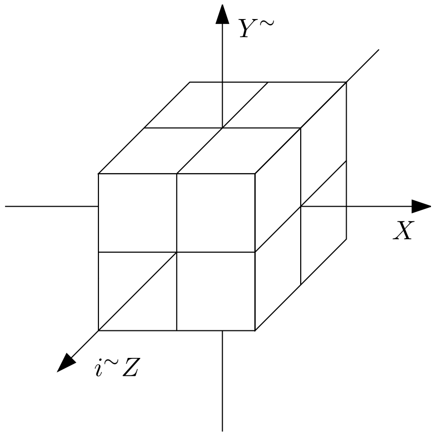

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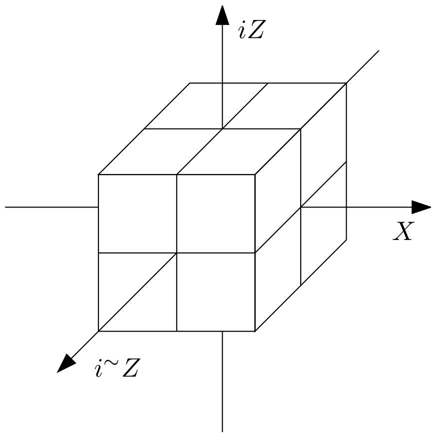

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[45]Ref. Sec. 4.3
[46]Ref. Sec. 4.3

Section: **The Subspaces of** S^4 (4.3 Page 69 [47])

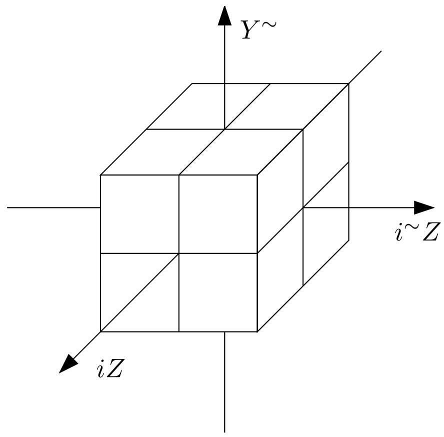

Figure 4.15: Cubicle 4 is the subspace $Y^{\sim}iZi^{\sim}Z$. Page 69

Section: **The Subspaces of** S^4 (4.3 Page 69 [48])

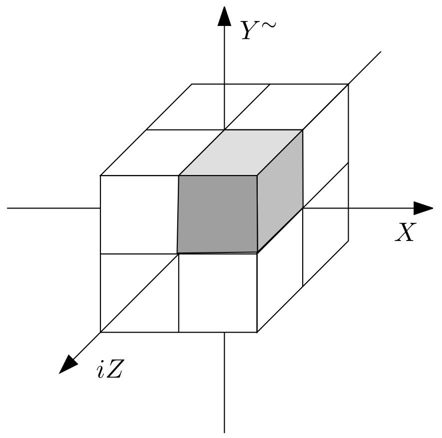

Figure 4.16: The octant 1 of the cubicle 1. Page 69

[47] Ref. Sec. 4.3
[48] Ref. Sec. 4.3

Chapter 5

Index of Nomenclature

A Content of Special Symbols (In order of appearance)

Chapter 1: Ordered Pairs

$\mathbb{R}$: The class of all real numbers, 2

$F = \{F; +, *\}$: A field with operations + and *, 2

$|a|$: The absolute value of the real number a, 4

$\mathbb{O}$: The class of all ordered pairs, 5

$\mathbb{R}^{\sim}$: The class of all image-real numbers, 6

(a, b) : The ordered pair of the real numbers a and b, 5

$(a, b)^{\sim}$: The image of the ordered pair (a, b), 5

$a^{\sim}$: The image of the real number a, 6

X**-axis** : The graphical representation of the real numbers class. An alias for $\mathbb{R}$. (See $\mathbb{R}$), 6

$Y^{\sim}$**-axis** : The graphical representation of the image-real numbers class. An alias for $\mathbb{R}^{\sim}$. (See $\mathbb{R}^{\sim}$), 6

$(0, 0) = 0 = O$: The common point of the axes in the coordinate system, 6

$d(a, b)$: The scalar product of d times (a, b), 12

$(-a, -b)$: The opposite of the ordered pair (a, b), 13

$\overline{(a, b)}$: The conjugate of the ordered pair (a, b), 13

$(a, b)^{-1}$: The inverse of the ordered pair (a, b), 13

$\frac{(a,b)}{(c,d)}$: Division of ordered pairs, 14

$|(a, b)|$: The absolute value of the ordered pair (a, b), 15

$\|(a, b)\|$: The norm of the ordered pair (a, b), 17

$Arg(a, b)$: The principal argument of the ordered pair (a, b), 19

$(a, b)^n$: Exponentiation of ordered pair (a, b) to the real number n, 20

$\sqrt{(a, b)}$: Radicalization of ordered pair (a, b), 19

Chapter 2: Complex Numbers

ai : The imaginary number ai, 24

$ai\tilde{}$: The image of the imaginary number $ai\tilde{}$, 24

i : The unitary imaginary number, 25

$i\tilde{}$: The image of the unitary imaginary number i, 25

$\mathbb{I}$: The class of all imaginary numbers, 25

$\mathbb{I}\tilde{}$: The class of all image-imaginary numbers, 25

(ai, bi) : An ordered pair of the imaginary numbers ai and bi, 26

$(ai, bi)\tilde{}$: An image-imaginary ordered pair. The image of (ai, bi), 26

$c(ai, bi)$: The scalar product of the real number c times an imaginary ordered pair, 30

$(a, b)i$: Imaginary scalar product of a real ordered pair times the unit imaginary number i, 31

$\big|(ai, bi)\big|$: The absolute value of an imaginary ordered pair, 31

$C = a + ci$: A complex number formed by the real number a and the imaginary number ci, 32

$C = (a, ci)$: The ordered pair representation of the complex number $C = a + ci$, 32

$Re(C)$: The real part of the complex number C, 33

$Im(C)$: The imaginary part of the complex number C, 33

$|C|$: The absolute value of the complex number C, 37

$\|C\|$: The norm of the complex number C, 37

$Arg(C)$: The principal argument of the complex number C, 39

Chapter 3: Transcomplex Numbers

$T = (a,b) + (c,d)i$: A transcomplex number, 44

$\mathbb{T}$: The class of all transcomplex numbers, 44

$Re(T)$: The real part of a transcomplex number T, 45

$Im(T)$: The imaginary part of a transcomplex number T, 45

$r(a,b,c,d)$: The scalar product of a real number r times a transcomplex number, 46

$|T|$: The absolute value of a transcomplex number T, 46

$\|T\|$: The norm of a transcomplex number T, 47

$Arg(T)$: The principal argument of a transcomplex number T, 48

$^{+}\|T\|$: The transnorm of a transcomplex number T, 49

$^{+}Arg(T)$: The transargument of a transcomplex number T, 49

T^n : Exponentiation of a transcomplex number T to a real exponent n, 51

$T^{T'}$: Transcomplex exponentiation of transcomplex number T to a another transcomplex T', 52

Chapter 4: The Coordinate System S^4

S^4 : The 4-dimensional space of the transcomplex numbers, 54

X**-coordinate** : The graphical representation of the element a of the transcomplex $T = (a, b, c, d)$. (See X-axis), 54

$\tilde{Y}$**-coordinate** : The graphical representation of the element b of the transcomplex $T = (a, b, c, d)$. (See $\tilde{Y}$-axis), 54

iZ**-coordinate** : The graphical representation of the element c of the transcomplex $T = (a, b, c, d)$, 54

$\tilde{\mathbf{i}}$Z-coordinate : The graphical representation of the element d of the transcomplex $T = (a, b, c, d)$, 54

$U \perp V$: Mutually perpendicular transcomplex numbers U and V, 11

$\mathbb{U} \perp \mathbb{V}$: Mutually orthogonal sets $\mathbb{U}$ and $\mathbb{V}$, 58

3/4 **space** : Any three axes representation of the space S^4, 65

Co#(*axis-name*) : The coordinate number of the axis *axis-name*, 65

$Q\#(V1, V2, V3)$: The cubicle number of the triaxial space $(V1, V2, V3)$, 66

Oct# **(*subcubicle-name*)** : Octant number of the subcubicle *subcubicle-name*, 68

$Ty\#(T)$: The number assigned to the type of transcomplex the number T, 70

Chapter 5: Transcomplex Functions

$A \xrightarrow{f} B$: A map (function or transformation) from a domain set A to a range set B, 73

f^{-1} : The inverse of the map f, 75

$(1-1)$: One-to-one map (function or transformation), 75

I : The identity map, 76

$g \circ f$: Product (or composition) of the map g and f, 76

$f^{\sim}$: The image of the map f, 79

$F(x+iz)$: A complex function (or map), 79

$F^{\sim}$: A semicomplex map, 82

$F*$: A transcomplex map, 83

SSSS : Transcomplex surface, 87

Index of Theorems

A Content of Theorems Stated (In order of appearance)

Chapter 1: Ordered Pairs

Theorem 1.1: (p. 10) Let a and b be any two unknown numbers belonging to any of the X or $Y^{\sim}$-axes, then $ab = 0$ if only if a belongs to X and b belongs to $Y^{\sim}$ or the converse. In symbols:

$$ab = 0$$

if and only if

$$a \in X \qquad \text{and} \qquad b \in Y^{\sim}$$

or

$$a \in Y^{\sim} \qquad \text{and} \qquad b \in X$$

where

$$X = \mathbb{R} \qquad \text{and} \qquad Y^{\sim} = \mathbb{R}^{\sim}.$$

Corollary. If the product of two real numbers is zero, then at least one of them is zero

Theorem 1.2: (p. 20) The class $\mathbb{O}$, that is, the class of all real numbers ordered pairs, together with the operations + and *, make a field,

Chapter 2: Complex Numbers

Theorem 2.1: (p. 37) The unit element under multiplication of the complex numbers field is the complex number $1 + 0i = 1$,

Theorem 2.2: (p. 37) The inverse of the complex number $C = a + ci = (a, ci)$ is the complex number C^{-1} given by:

$$\begin{aligned} C^{-1} &= \frac{a}{a^2 + c^2} - \frac{c}{a^2 + c^2} \\ &= \Big(\frac{a}{a^2 + c^2}, -\frac{c}{a^2 + c^2}\Big). \end{aligned}$$

Theorem 2.3: (p. 38) Let $C = a + ci$ and $C' = a' + c'i$ be any two complex numbers, then

$$|C| * |C'| \neq |C * C'|$$

but on the contrary,

$$\|C\| * \|C'\| = \|C * C'\|.$$

Theorem 2.4: (p. 39) Let $C = a + ci$, then

$$\|C^{-1}\| = \|C\|^{-1}$$

and

$$Arg(C^{-1}) = -Arg(C).$$

Theorem 2.5: (p. 40) Let $C = a + ci$ and $C' = a' + c'i$ be any two complex numbers. Then:

$$Arg(C * C') = Arg(C) + Arg(C')$$

and

$$Arg\Big(\frac{C}{C'}\Big) = Arg(C) - Arg(C').$$

Theorem 2.6: (p. 41) Let C and C' be two complex numbers expressed in trigonometric form.

$$C = \|C\|\Big(cos(Arg(C), sin(Arg(C))i\Big)$$

and

$$C' = \|C'\|\Big(cos(Arg(C'), sin(Arg(C'))i\Big).$$

Also let

$$K = \|C\| * \|C'\|$$

and

$$L = Arg(C) + Arg(C')$$

then

$$C * C' = K\Big(cos(L) + sin(L)i\Big).$$

Chapter 3: Transcomplex Numbers

Theorem 3.1: (p. 44) Two transcomplex numbers $T = (a, b, c, d)$ and $T' = (a', b', c', d')$ are equal if and only if $a = a'$, $b = b'$, $c = c'$, and $d = d'$.

Theorem 3.2: (p. 45) Let T be any transcomplex number $T = (a, b, c, d)$, then

$$T = a + b^{\tilde{}} + ci + (di)^{\tilde{}}.$$

Theorem 3.3: (p. 47) Let $T = (a, b, c, d)$ and $T' = (a', b', c', d')$. Then

$$\|T\| * \|T'\| = \|T * T'\|.$$

Theorem 3.4: (p. 48) Let $T = (a, b, c, d)$ and $T' = (a', b', c', d')$. Then

$$Arg(T * T') = Arg(T) + Arg(T').$$

Theorem 3.5: (p. 48) Let $T = (a, b, c, d)$ be a transcomplex number. Then

$$T = \|T\| * \Big(cos\big(Arg(T)\big), sin\big(Arg(T)\big)\Big).$$

Theorem 3.6: (p. 50) Let $T = (a, b, c, d)$ and $T = (a', b', c', d')$. Then,

$$^{+}\|T\| \; * \; ^{+}\|T'\| \; \neq \; ^{+}\|T * T'\|.$$

Theorem 3.7: (p. 50) Let $T = (a, b, c, d)$ and $T = (a', b', c', d')$. Then,

$$^{+}Arg(T) \; * \; ^{+}Arg(T') \neq \; ^{+}Arg(T) +^{+} Arg(T').$$

Theorem 3.8: (p. 50) Let $T = (a, b, c, d)$ be a transcomplex number. For short, let

$$R =^{+} \Big\|\|T\|\Big\| = (r_1, r_2)$$

and

$$W =^{+} Arg(T) = \big\|Arg(T)\big\| = (\Theta_1, \Theta_2)$$

then

$$T = Re^{iW}$$

where e is the natural logarithm base and i the imaginary unit.

Theorem 3.8: (p. 52) The transcomplex number class $\mathbb{T}$, together with the addition and multiplication operations defined for this class make a field.

Chapter 4: The Coordinate System S^4

Theorem 4.1: (p. 58) Respect to the orthogonality of the axes of the space S^4 the following two propositions are true:

- The four axes of the tetraspace S^4 are mutually orthogonal.
- No axis is orthogonal to itself.

Theorem 4.2: (p. 58) No element of one axis can be plotted into another axis.

Chapter 5: Transcomplex Functions

Theorem 5.1: (p. 84) Let F^*be a transcomplex map from a domain A of complex variables $U = (x, 0, z)$ to a range B of transcomplex numbers $W = (x', y', z')$. Then

$$F^*(x, z) = (x, f(x, z), g(x, z)).$$

Theorem 5.2: (p. 85) A transcomplex map $F*(U)$ is a semicomplex map $F\tilde{}(U)$ plotted in the plane $X = x$. In symbols,

$$F*(U) = x + F\tilde{}(U).$$

Theorem 5.3: (p. 86) If the domain D of a transcomplex function is of real numbers only, then that transcomplex function reduces to its simple real function expression. That is:

$$F^*(U) = f(x)$$

when $x \in \mathbb{R}$.

Theorem 5.4: (p. 88) Let $U = (x, z)$ be a complex region on the complex plane and let $F(U) = f(x, z) + g(x, z)i$ be the complex function defined on the region U. Then the interception of the transcomplex map $F*(U)$ with the $XY\tilde{}$plane is the same as the image-real map $f(x)$ defined for the subdomain of U of all ordered pair such that $z = 0$

Foundations Of Transcomplex Numbers

An extension of the complex number system to four dimensions

This book is about the complex numbers: their foundations, their operations. But the final goal is the plotting of the complex variables. Instead of the usual approach of using two different Cartesian planes, where one plane is used for the domain of the function and another plane is used for its range, this book develops a consistent foundation for a new 4 axes coordinate space where the domain and the range are part of the same structure.

This new number structure of complex numbers and complex variables is accompanied with a few surprises like the one that the multiplication of two transcomplex numbers can sometimes be zero, even when none of the factors is zero. The visualization of the behavior of functions of complex variables is unique and surprising: the complex quadratic function is a saddle-like surface, and the exponential surface turns out to be something similar to the Torricelli's trumpet.

As presented here, the complex numbers are not just a pair of a real and an imaginary number; complex number are entities defined by its properties and operations; therefore the "ordinary" complex numbers are just a subfield of a more general field of complex entities: the transcomplex numbers.

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